MTH 122 – Spring 2004
EXAM 3-A-Solutions

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You have 60 minutes. Answer 4 questions out of 1– 6 (each one is worth 20 points), and answer questions 7 and 8. Mark clearly which questions are **not** to be graded. Show full logic and work for full credit. **Good luck!**

In this exam a and b are known constants!

1. Find and sketch the asymptotes of the function

$$y = \frac{5+x}{1+2x}.$$

A: The vertical asymptotes are where the denominator vanishes, thus

$$x = -\frac{1}{2}, \qquad (4pts)$$

and then

$$\lim_{x\to -1/2-} \frac{5+x}{1+2x} = -\infty, \qquad \lim_{x\to -1/2+} \frac{5+x}{1+2x} = \infty. \qquad (6pts)$$

The horizontal asymptote is the y-value when x tends to infinity or negative infinity, thus

$$y = \frac{1}{2}, \qquad (4pts)$$

since

$$\lim_{x\to \pm\infty} \frac{5+x}{1+2x} = \lim_{x\to -\infty} \frac{\frac{5}{x}+1}{\frac{1}{x}+2} = \frac{1}{2}. \qquad (6pts)$$

2. Find the indefinite integral

$$\int \frac{b}{e^{-x/a}}\,dx.$$

A: First, we rewrite the integral as $b\int e^{x/a}\,dx$. Then we let $u = x/a$ and so

$$u' = 1/a, \qquad du = (1/a)dx, \qquad \text{and} \quad dx = adu, \qquad\qquad (8pts)$$

and the integral is

$$b\int e^{x/a}\,dx = ab\int e^{u}\,du = abe^{u} + C \qquad\qquad (8pts)$$

$$= abe^{x/a} + C. \qquad\qquad (4pts)$$

You may also choose $u = -x/a$ or even $u = e^{-x/a}$, but using either one is a bit longer.

3. Find the definite integral

$$\int_1^4 \frac{\ln x}{x}\,dx.$$

A: First, we need to find the antiderivative $F(x)$ of $\ln x/x$.

To this end let $u = \ln x$ then

$$u' = \frac{1}{x}, \qquad du = \frac{1}{x}dx, \qquad\qquad (6pts)$$

and then the integral of $\ln x/x$ can be written as

$$F(x) = \int \frac{\ln x}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln|x|)^2 + C. \qquad\qquad (8pts)$$

Using now the Fundamental Theorem of Calculus we have

$$\int_1^4 \frac{\ln x}{x}\,dx = F(4) - F(1) = \frac{1}{2}(\ln 4)^2 = .96. \qquad\qquad (6pts)$$

Here, we used the fact that $\ln 1 = 0$.

4. Find the definite integral

$$\int_0^b \frac{1}{x+a}\,dx.$$

A: First, we need to find the antiderivative $F(x)$ of $1/(x+a)$.
To this end let $u = x + a$ then

$$u' = 1, \qquad du = dx, \qquad\qquad (6pts)$$

and then the integral of $1/(x+a)$ can be written as

$$F(x) = \int \frac{1}{x+a}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|x+a| + C. \qquad (8pts)$$

Using now the Fundamental Theorem of Calculus we have

$$\int_0^b \frac{1}{x+a}\,dx = F(b) - F(0) = \ln|b+a| - \ln|a| = \ln\left|\frac{b+a}{a}\right|. \qquad (6pts)$$

5. Find the indefinite integral

$$\int \frac{2xe^{x^2}}{e^{x^2}+a}\,dx.$$

A: We choose

$$u = e^{x^2} + a, \qquad\qquad (4pts)$$

and then

$$u' = 2xe^{x^2}, \qquad du = 2xe^{x^2}\,dx. \qquad\qquad (8pts)$$

Therefore,

$$F(x) = \int \frac{2xe^{x^2}}{e^{x^2}+a}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|e^{x^2}+a| + C. \qquad (8pts)$$

6. The cost per hour (in dollars) of operating a certain car is given by

$$C = 0.4s - 0.004s^2 + 0.1, \qquad \text{for } \ 0 \le s \le 60,$$

where s is the speed in mph. At what speed is the cost C minimum? Maximum?

A: We need to find the critical points of the function C. We have

$$C' = 0.4 - 0.008s,$$

and $C' = 0$ when $s = 50$ and, since $C'' = -0.008 < 0$, the function is concave down (12 *pts*). Next,

$$C(50) = 10.1, \qquad C(0) = 0.1, \qquad C(60) = 9.7 \qquad\qquad (6\ pts)$$

We conclude that the cost in minimum when $s = 0$, and is maximum when $s = 50$. (2 *pts*)

7. (You have to answer this question.) A manufacturer has to design a container in the form of a rectangular box with a square base, no top, and with volume $V = 108$. What are the dimensions of the box that require the least amount of material?

A: Since the base of the box is a square, if we denote its length by$x$ and the height of the box by y we have that the volume V of the box is

$$V = x^2y = 108.$$

Therefore, $y = 108/x^2$ (2 *pts*). Next, the area of the box consists of the basis, x^2 and four sides with area xy each. Thus, the area S of the box is

$$S = x^2 + 4xy = x^2 + 4x\frac{108}{x^2} = x^2 + \frac{432}{x}. \qquad (3\ pts)$$

To find the minimum we find the critical points, so

$$S' = 2x - \frac{432}{x^2}, \qquad S'' = 2 + \frac{864}{x^3} > 0. \qquad (3\ pts)$$

Then $S' = 0$ at $x = 6$, and it is a minimum point since the function is concave up. We conclude that the dimensions of the desired box are $x = 6$ and $y = 3$. (2 *pts*)

8. (You have to answer this question.) A manufacturer's marginal-cost function is

$$\frac{dc}{dq} = 0.3q^2 - 0.4q + 10.$$

Determine the cost involved to increase production from 60 to 70 units.

A: We are given the marginal-cost function $c'(q)$, which is the derivative of the cost function $c(q)$, and asked about the cost involved in increasing q from 60 to 70, that is we need to compute $c(70)-c(60)$. Thus, we need to calculate

$$c(70) - c(60) = \int_{60}^{70} \frac{dc}{dq}\, dq = \int_{60}^{70} \big(0.3q^2 - 0.4q + 10\big)\, dq. \qquad (2\ pts)$$

First, we find the antiderivative, c,

$$\begin{aligned} c(q) &= \int \big(0.3q^2 - 0.4q + 10\big)\, dq \\ &= 0.3\int q^2\, dq - 0.4\int q\, dq + 10\int\, dq. \qquad (2\ pts) \end{aligned}$$

Now, we have (here C is the constant of integration),

$$0.3\int q^2\, dq = 0.3\frac{1}{3}q^3 + C = 0.1q^3 + C, \qquad (1\ pt)$$

$$0.4\int q\, dq = 0.4\frac{1}{2}q^2 + C = 0.2q^2 + C, \qquad (1\ pt)$$

$$10\int\, dq = 10q + C. \qquad (1\ pt)$$

Thus,

$$c(q) = 0.1q^3 - 0.2q^2 + 10q + C. \qquad (3\ pts)$$

Finally, the answer is

$$\begin{aligned} c(70) - c(60) &= \big(0.1(70)^3 - 0.2(70)^2 + 10(70)\big) \\ &\quad - \big(0.1(60)^3 - 0.2(60)^2 + 10(60)\big) \\ &= 34,020 - 21,480 = 12,540. \qquad (2\ pts) \end{aligned}$$