

# APM-541 Fall 2003

## Final EXAM – B

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You have 180 minutes. Answer 10 questions out of 1–13, and answer questions 14 and 15. Mark clearly which questions are **not** to be graded. Each question is worth 16 points (total of 200). You may use a photocopy of the tables of Laplace and Fourier Transforms, and two two-sided pages freely written. Please attach the freely written pages to the exam. Show full logic for full credit.

**Good luck!**

1. Use the convolution theorem to find the inverse transform of

$$\frac{1}{(s-1)(s+2)}.$$

A: We write it as  $H(s)G(s)$ , where  $H(s) = 1/(s-1)$  and  $G(s) = 1/(s+2)$ . Then,

$$\mathcal{L}^{-1}(H(s)) = e^t, \quad \mathcal{L}^{-1}(G(s)) = e^{-2t},$$

and by the convolution theorem

$$\begin{aligned} \mathcal{L}^{-1}(H(s)G(s)) &= \int_0^t e^{-2\tau} e^{(t-\tau)} d\tau = e^t \int_0^t e^{-2\tau} e^{-\tau} d\tau \\ &= e^t \int_0^t e^{-3\tau} d\tau = \frac{1}{3} e^t [-e^{-3\tau}]_0^t \\ &= \frac{1}{3} e^t (1 - e^{-3t}). \end{aligned}$$

2. What is the rank of the matrix

$$\begin{pmatrix} 0 & 1 & a \\ 1 & 0 & 1 \\ 1 & b & 0 \end{pmatrix}.$$

How does it depend on  $a, b$ ? Can you write down a basis for the row space?

A: By using elementary row operations we find

$$\begin{pmatrix} 0 & 1 & a \\ 1 & 0 & 1 \\ 1 & b & 0 \end{pmatrix} \implies \begin{pmatrix} 0 & 1 & a \\ 1 & 0 & 1 \\ 0 & b & -1 \end{pmatrix} \implies \begin{pmatrix} 0 & 1 & a \\ 1 & 0 & 1 \\ 0 & 0 & 1+ab \end{pmatrix},$$

where the last matrix was obtained by multiplying the first row by  $-b$  adding to the last row and multiplying the result by  $-1$ . Hence, if  $ab \neq -1$  the rank is 3, and a basis is  $\{(0, 1, a), (1, 0, 1), (0, 0, 1)\}$ .

If  $ab = -1$  then the rank is 2 and a basis is  $\{(0, 1, a), (1, 0, 1)\}$ .

3. State a condition which guarantees that the linear system  $A\mathbf{x} = \mathbf{b}$  has a solution. When does it have many solutions? How many?

A: A condition for solvability is that

$$\text{rank}A = \text{rank}\tilde{A},$$

where  $\tilde{A}$  is the augmented matrix.

The solution is unique if

$$\text{rank}A = \text{rank}\tilde{A} = n.$$

There is infinity of solutions if

$$\text{rank}A = \text{rank}\tilde{A} < n.$$

Another condition is that  $A$  is invertible and then the unique solution of the system is given by  $\mathbf{x} = A^{-1}\mathbf{b}$ .

4. Compute the work done by the force  $\mathbf{F} = (x - y)^2\mathbf{i} + (xy)^5\mathbf{j}$  along the curve  $C : xy = 1$ , for  $2 \leq x \leq 4$ . A: We have that  $y = 1/x$  on  $C$ , and so

$$\mathbf{F} = \left(x - \frac{1}{x}\right)^2\mathbf{i} + \mathbf{j} = \left(x^2 - 2 + \frac{1}{x^2}\right)\mathbf{i} + \mathbf{j}, \quad \text{on } C.$$

Now, let  $t = x$  be the curve parameter, then  $2 \leq t \leq 4$  and

$$C : \quad \mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}, \quad \text{and} \quad \mathbf{r}'(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}.$$

Thus,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \mathbf{F} \cdot \mathbf{r}' dt = \int_2^4 \left(\left(t^2 - 2 + \frac{1}{t^2}\right)\mathbf{i} + \mathbf{j}\right) \cdot \left(\mathbf{i} - \frac{1}{t^2}\mathbf{j}\right) dt \\ &= \int_2^4 (t^2 - 2) dt = \left(\frac{1}{3}t^3 - 2t\right)\Big|_2^4 = \frac{64}{3} - 8 - \frac{8}{3} + 4 = \frac{44}{3}. \end{aligned}$$

5. Choose a curve  $C$  connecting  $(0, 0, 0)$  to  $(a, b, c)$ , and compute the integral

$$\int_C (2e^{2z}\mathbf{i} + 12y\mathbf{j} + 4xe^{2z}\mathbf{k}) \cdot d\mathbf{r}.$$

A: One can compute the integral directly, by choosing  $\mathbf{r} = at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k}$ , for  $0 \leq t \leq 1$ , which is the straight line connecting the points, however, this involves integration by parts. A simpler way is to note that

$$f = f(x, y, z) = 2xe^{2z} + 6y^2 \quad \longrightarrow \quad \nabla f = 2e^{2z}\mathbf{i} + 12y\mathbf{j} + 4xe^{2z}\mathbf{k},$$

and, therefore, the integral is given by

$$I = f\Big|_{(0,0,0)}^{(a,b,c)} = 2ae^{2c} + 6b^2.$$

6. Given a region  $D$  is space with surface  $S$ . How is the surface integral

$$J = \frac{1}{2} \int \int_S (x dy dz + z dx dy)$$

related to the volume of  $D$ .

A: We note that

$$\int \int_S \mathbf{F} \cdot \mathbf{n} dA = \int \int_S (F_1 dy dz + F_2 dx dz + F_3 dx dy).$$

Therefore, we must have  $F_2 = 0$ , so if we choose  $\mathbf{F} = x\mathbf{i} + z\mathbf{k}$ , then  $\text{div } \mathbf{F} = 2$ , and it follows from the Gauss Divergence Theorem that

$$\begin{aligned} J &= \frac{1}{2} \int \int_S (x dy dz + z dx dy) = \frac{1}{2} \int \int_S (x\mathbf{i} + z\mathbf{k}) \cdot \mathbf{n} dA \\ &= \frac{1}{2} \int \int \int_D \text{div} (x\mathbf{i} + z\mathbf{k}) dV = \int \int \int_D dV = \text{Vol}(D). \end{aligned}$$

We conclude that  $J = \text{Vol}(D)$ .

7. Use the Laplace transform to solve for  $y$ , where  $r = r(t) = 1$  if  $0 < t < 1$  and  $r = 0$  otherwise,  $y(0) = 1$ ,  $y'(0) = 0$ , and  $y'' + 16y = r(t)$ .

A: Let  $Y = Y(s) = \mathcal{L}(y)$ , and note that  $r(t) = 1 - u(t - 1)$ , so that  $R = R(s) = \mathcal{L}(r) = (1 - e^{-s})/s$ . Applying the Laplace transform to the equation and using the initial conditions yields

$$(s^2 + 16)Y = \frac{1}{s} - \frac{e^{-s}}{s} + s.$$

Thus

$$Y = \frac{1}{s(s^2 + 16)} - e^{-s} \frac{1}{s(s^2 + 16)} + \frac{s}{s^2 + 16}.$$

From the tables the third term yields  $\cos(4t)$ ; while the first term is

$$\frac{1}{s(s^2 + 16)} = \frac{1}{16s} - \frac{s}{16(s^2 + 16)}.$$

Thus, using the tables

$$\mathcal{L}^{-1} \left( \frac{1}{s(s^2 + 16)} \right) = \frac{1}{16} - \frac{1}{16} \cos(4t).$$

Finally, collecting all the terms yields

$$y(t) = \cos(4t) + \frac{1}{16}(1 - \cos(4t)) - \frac{1}{16}(1 - \cos(4(t - 1)))u(t - 1).$$

8. Find  $\int \int_S \mathbf{F} \cdot \mathbf{n} dA$ , the total outflux of the vector field  $\mathbf{F} = 12\mathbf{i} + 4y\mathbf{j} + 8x^2y\mathbf{k}$  over the surface  $S$  that encloses the hemisphere  $x^2 + y^2 + z^2 = 25$  and  $0 \leq z$ .

A: We use the Gauss Divergence Theorem, so we compute  $\text{div } \mathbf{F} = 4$ . Then,

$$\begin{aligned} \int \int_S \mathbf{F} \cdot \mathbf{n} dA &= \int \int \int_D \text{div } \mathbf{F} dV = 4 \int \int \int_D dV \\ &= 4 \text{Vol}(D) = 4 \frac{2\pi}{3} 5^3 = \frac{1000}{3} \pi, \end{aligned}$$

since the volume of the half-hemisphere is  $2\pi r^3/3$  and  $r = 5$ .

9. A particle with mass  $m$  is moving with speed  $vt$  in the direction of the unit vector  $\mathbf{b}$ , which is rotating with constant angular velocity  $\omega$ .

Find the force that is acting on the particle and its acceleration vector.

A: The particle velocity vector is

$$\mathbf{v}(t) = \mathbf{r}'(t) = vt\mathbf{b},$$

and NOT that the position vector is  $\mathbf{r}(t) = vt\mathbf{b}$ ! Hence,  $\mathbf{a}(t) = \mathbf{v}'(t) = v\dot{\mathbf{b}} + vt\dot{\mathbf{b}}$ . Next,

$$\mathbf{b} = \cos \omega t \mathbf{i} + \sin \omega t \mathbf{j}.$$

Thus,

$$\mathbf{v}(t) = vt \cos \omega t \mathbf{i} + vt \sin \omega t \mathbf{j}.$$

The acceleration is  $\mathbf{v}'$ , so

$$\begin{aligned} \mathbf{a} = \mathbf{v}' &= v(\cos \omega t \mathbf{i} + \sin \omega t \mathbf{j}) + vt\omega(-\sin \omega t \mathbf{i} + \cos \omega t \mathbf{j}) \\ &= v(\cos \omega t - \omega t \sin \omega t) \mathbf{i} + v(\sin \omega t + \omega t \cos \omega t) \mathbf{j}. \end{aligned}$$

The force that is acting on the particle is  $\mathbf{F} = m\mathbf{a}$ .

10. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}.$$

When are they linearly independent?

A: To find the eigenvalues we compute the determinant

$$\text{Det} \begin{pmatrix} 1 - \lambda & a \\ 0 & b - \lambda \end{pmatrix} = (1 - \lambda)(b - \lambda) = 0,$$

and we see that the eigenvalues are  $\lambda_1 = 1$  and  $\lambda_2 = b$ . An eigenvector  $(x, y)^T$  that corresponds to  $\lambda_k$ , for  $k = 1, 2$  is obtained by solving

$$(1, a) \cdot (x, y) = \lambda_k x.$$

Thus, for  $\lambda_1 = 1$  we find that  $x + ay = x$ , hence  $y = 0$ , unless  $a = 0$  and then  $y$  is arbitrary so we may choose it as  $y = 0$ , and an eigenvector is  $\mathbf{v}_1 = (1, 0)$ .

For  $\lambda_2 = b$  we find that

$$x + ay = bx,$$

hence we have the following cases:

(a)  $b \neq 1$ ,  $a \neq 0$ , and then  $y = (b - 1)x/a$  and an eigenvector is  $\mathbf{v}_2 = (1, (b - 1)/a)$ ;

(b)  $b \neq 1$ ,  $a = 0$ , and then  $x = bx$  so that  $x = 0$  and an eigenvector is  $\mathbf{v}_2 = (0, 1)$ ;

(c)  $b = 1$ ,  $a = 0$ , and then  $x = x$  so that  $x$  is arbitrary and an eigenvector is  $\mathbf{v}_2 = (1, 1)$ ;

(d)  $b = 1$ ,  $a \neq 0$ , and then  $x + ay = x$  so that  $y = 0$  and the eigenvector is the same as for  $\lambda_1$ , so we have only one eigenvector.

11. Compute the Fourier Series of the periodic function

$$f(x) = \begin{cases} 2 & \text{if } 0 < x < \pi/2, \\ 0 & \text{if } \pi/2 < x < 2\pi. \end{cases}$$

A: We have to compute  $a_0, a_n, b_n$ , for  $n = 1, 2, 3, \dots$  in the series. So,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{2\pi} \int_0^{\pi/2} dx = \frac{1}{\pi} \frac{\pi}{2} = \frac{1}{2}.$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi/2} \cos nx \, dx \\
&= \frac{2}{\pi} \frac{\sin nx}{n} \Big|_0^{\pi/2} = \frac{2}{\pi} \frac{\sin n\pi/2}{n} = \begin{cases} 0 & \text{if } n = 2m, \\ \frac{2}{\pi n} & \text{if } n = 4m + 1, \\ \frac{-2}{\pi n} & \text{if } n = 4m + 3. \end{cases}
\end{aligned}$$

Next,

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi/2} \sin nx \, dx = \\
&= \frac{-2}{\pi} \frac{\cos nx}{n} \Big|_0^{\pi/2} = \frac{-2}{n\pi} (\cos n\pi/2 - 1) = \begin{cases} 0 & \text{if } n = 2m + 1, \\ \frac{-2}{\pi n} & \text{if } n = 4m, \\ \frac{2}{\pi n} & \text{if } n = 4m + 2. \end{cases}
\end{aligned}$$

We conclude that

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \cos x + \frac{1}{\pi} \sin 2x - \frac{2}{3\pi} \cos 3x - \frac{1}{2\pi} \sin 4x \dots$$

12. Evaluate the surface integral of the normal component of the function  $\mathbf{F} = xy^2\mathbf{i} + 12y\mathbf{j} - y^2z\mathbf{k}$  over the surface of the tetrahedron with vertices  $(0, 0, 0)$ ,  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$ .

A: We use the Gauss Divergence Theorem. First,

$$\operatorname{div} \mathbf{F} = \operatorname{div}(xy^2\mathbf{i} + 2y\mathbf{j} - y^2z\mathbf{k}) = y^2 + 12 - y^2 = 12.$$

Then

$$\begin{aligned}
\int \int_S \mathbf{F} \cdot \mathbf{n} \, dA &= \int \int \int_D \operatorname{div} \mathbf{F} \, dV = 12 \int \int \int_D dV \\
&= 12 \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz \, dy \, dx = 6 \int_0^1 (1 - 2x + x^2) \, dx = \\
&= 6(x - x^2 + \frac{1}{3}x^3) \Big|_0^1 = \frac{6}{3} = 2.
\end{aligned}$$

13. Find the Fourier Transform of the function

$$f(x) = \begin{cases} -1 & -a < x < 0 \\ 1 & 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

A: We could use twice item 2 in Table III, but we can also do it directly,

$$\begin{aligned} \mathcal{F}(f) = \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx = \\ &= \frac{1}{\sqrt{2\pi}} \left( - \int_{-a}^0 e^{-i\omega x} dx + \int_0^a e^{-i\omega x} dx \right) = \\ &= \frac{1}{\sqrt{2\pi}} \left( \frac{1}{i\omega}(1 - e^{i\omega a}) + \frac{1}{i\omega}(1 - e^{-i\omega a}) \right) = \\ &= \frac{1}{i\omega\sqrt{2\pi}} (2 - e^{i\omega a} - e^{-i\omega a}) = \frac{2}{i\omega\sqrt{2\pi}} (1 - \cos \omega a). \end{aligned}$$

Here, we used Euler's formula  $e^{\pm i\omega a} = \cos \omega a \pm i \sin \omega a$ .

14. (You have to answer this question.) The flow of a fluid in space is described by the velocity field  $\mathbf{v} = x\mathbf{i} + y^2\mathbf{j}$ . Is the flow compressible?

Describe the path of a particle which is initially at  $(1, 1, 1)$ . Where will the particle be at  $t = 1/2$ , and  $t = 2$ ? A: To check compressibility we compute the divergence

$$\operatorname{div} \mathbf{v} = \operatorname{div} (x\mathbf{i} + y^2\mathbf{j}) = 1 + 2y,$$

hence the flow is compressible.

To find the trajectories of particles in the flow we note that if the flow is steady and the trajectory is  $\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ , then on the trajectory  $\mathbf{v} = \mathbf{r}'$ , thus

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = y^2, \quad \frac{dz}{dt} = 0.$$

We rewrite the differential equations as

$$\frac{dx}{x} = dt, \quad \frac{dy}{y^2} = dt, \quad \frac{dz}{dt} = 0,$$

and integrate, which yields

$$\ln x = t + C, \quad \frac{-1}{y} = t + C, \quad z(t) = z_0.$$

Therefore, taking into account that initially at  $t = 0$  the particle is at the point  $(x_0, y_0, z_0)$ , we find

$$x(t) = x_0 e^t, \quad y(t) = \frac{-1}{t - \frac{1}{y_0}}, \quad z(t) = z_0.$$

The trajectory is

$$\mathbf{r}(t) = x_0 e^t \mathbf{i} - \frac{1}{t - \frac{1}{y_0}} \mathbf{j} + z_0.$$

If initially  $x_0 = 1, y_0 = 1$  and  $z_0 = 1$  then at  $t = 1/2$

$$\mathbf{r}(1/2) = e^{1/2} \mathbf{i} - \frac{1}{\frac{1}{2} - 1} \mathbf{j} + 1,$$

and the particle is at the point  $(e^{1/2}, 2, 1)$ .

When  $t = 2$  we have

$$\mathbf{r}(2) = e^2 \mathbf{i} - \frac{1}{2 - 1} \mathbf{j} + 1,$$

and the particle is at the point  $(e^2, -1, 1)$ . **Whoops!** The particle is moving in the positive  $y$ -direction, how did it get to  $y = -1$ ?

15. (You have to answer this question.) Use the Divergence Theorem to compute the outflux integral  $\int \int_S \mathbf{F} \cdot \mathbf{n} \, dA$  of the vector field  $\mathbf{F} = y^3 \mathbf{i} + 4x^2 \mathbf{j} + z \mathbf{k}$  over the surface  $S$  that encloses the semi-cone  $x^2 + y^2 = z$  and  $0 \leq y, 0 \leq z \leq 16$ .

A: We have  $\text{div} \mathbf{F} = \text{div}(y^3 \mathbf{i} + 4x^2 \mathbf{j} + z \mathbf{k}) = 1$ . Therefore, the surface integral equals the volume of the semi-cone, which in polar coordinates  $x = r \cos \theta, y = r \sin \theta, z = z$  can be written as

$$\text{Vol} = \int_0^{16} \int_0^{\sqrt{z}} \int_0^\pi d\theta r \, dr \, dz = \pi \int_0^{16} \left(\frac{1}{2} r^2\right) \Big|_0^{\sqrt{z}} dz = \frac{\pi}{2} \int_0^{16} z \, dz = 64\pi.$$