A Wave Equation arising in a 1D Piezoelectric System

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1 Introduction

We derive in this note the equation of motion of a one-dimensional rod made of a piezoelectric material. First the general equations of evolution for a body made of a piezoelectric material are given in Section 2, and then we show in Section 3 how to obtain a model for the 1D piezoelectric rod.

Piezoelectricity was discovered by the Curie brothers in 1880, and models the observation that certain crystals generate electric potential when mechanical loads are applied, and, conversely, generate mechanical stress when an electric potential is applied. In the first case they are often used as *sensors*, and in the second case as *actuators*. Such devices are ubiquitous these days and can be found in many of applications, such as the keyboard used to type these notes, or the headphones one uses to listen to music.

2 General Piezoelectricity

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain occupied by the piezoelectric body. We denote by $x_1 = x,\ x_2 = y,\ x_3 = z$ the space variables; $\boldsymbol{u} = (u_1, u_2, u_3)$ denotes the *displacements* vector field, $\boldsymbol{D} = (D_1, D_2, D_3)$ the *electric displacement* vector and $\boldsymbol{E} = (E_1, E_2, E_3)$ the *electric field*. In the static approximation the electric field is given by $\boldsymbol{E} = -\nabla\varphi$, where φ is the *electric potential*. We denote by $\varepsilon = (\varepsilon_{ij})$ the (linearized) *strain tensor* given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.1}$$

where here and below $i, j, k, \ell = 1, 2, 3$. The *stress tensor* is denoted by $\sigma = (\sigma_{ij})$. We also let $H = h_{kij})$ be the third-order tensor of *piezoelectric coefficients* and $\beta = (\beta_{ij})$ be the tensor of *dielectric permitivities*. Also, $A = (a_{ijkl})$ is the fourth-order tensor of *elastic compliances*, θ denotes the *temperature* (measured from a fixed temperature T_0), $\alpha = (\alpha_{ij})$ the tensor of the *coefficients of thermal expansion* and $\boldsymbol{p} = (p_i)$ the vector of *pyroelectric coefficients*, c is the *heat capacity*, and ρ is the material *density*.

The constitutive equations of a piezoelectric material are as follows.

$$\varepsilon_{ij} = a_{ijkl}\sigma_{kl} + h_{kij}E_k + \alpha_{ij}\theta, \tag{1.2}$$

$$D_k = h_{kij}\sigma_{ij} + \beta_{ki}E_i + p_i\theta, \tag{1.3}$$

$$S = \alpha_{ij}\sigma_{ij} + p_iE_i + \frac{c}{T_0}\theta. \tag{1.4}$$

Here, S is the system's enthropy. Summation over a repeated index is implied here and below.

The evolution equations in the electrostatic approximation for $\boldsymbol{u}$, φ and θ are:

$$\rho\frac{\partial^2 u_i}{\partial t^2} - \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \tag{1.5}$$

$$\nabla \cdot \boldsymbol{D} = \frac{\partial D_i}{\partial x_i} = 0, \tag{1.6}$$

$$\frac{\rho c}{T_0}\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial x_i}\left(\frac{k_{ij}}{T_0}\frac{\partial \theta}{\partial x_j}\right) = p_i\frac{\partial^2 \varphi}{\partial x_i \partial t} - \lambda_{ij}\frac{\partial^2 u_i}{\partial x_j \partial t}. \tag{1.7}$$

These equations, together with the constitutive relations (1.2)–(1.4) and the initial and boundary conditions lead to the problems of thermopiezoelectricity.

3 One-dimensional Systems

In this section we derive a model for a long thin rod which has one end fixed, while the other end is free, and an electric potential is applied to the ends.

We assume that the system is isothermal, so that $\theta = T_0$ and we need not consider the equation for the temperature.

The setting is depicted in Fig. 1.

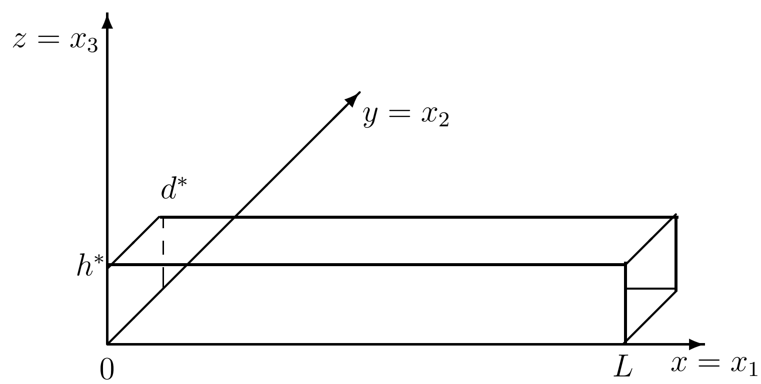

Figure 1. The setting of the problem.

We assume that the rod is thin and long, so the hight h^* and the thickness d^* satisfy and $h^*, d^* \ << L$. The left end ($x = 0$) and the right end ($x = L$) are conductors and the piezoelectric material occupies the domain

$$\Omega = \{(x_1, x_2, x_3) : \ 0 < x_1 < L, \ 0 < x_2 < d^*, \ 0 < x_3 < h^*\}.$$

Since $h^*, d^* \ << L$ we may assume that all the quantities depend on $x = x_1$ only. Since the left and right ends are conductors $E_2 = E_3 = 0$ there and we assume that this holds in $\overline{\Omega}$. Moreover, we assume that $u_2 = u_3 \approx 0$ and only $\sigma_{11} = \sigma \neq 0$.

We use the notation $x_1 = x, \ u_1 = u(x,t), \ E_1 = E(x,t)$ and $D_1 = D(x,t)$. Then, the constitutive equations (1.2)–(1.4) are

$$\varepsilon_{11} = \varepsilon = u_x = a\sigma + hE, \tag{2.1}$$

$$D_1 = D = h\sigma + \beta E. \tag{2.2}$$

It follows from (2.1) that

$$\sigma = \frac{1}{a} u_x - \frac{h}{a} E. \tag{2.3}$$

Now, (1.6) and (2.2) imply that

$$D_x = h\sigma_x + \beta E_x = 0.$$

Hence, $E_x = -(h/\beta)\sigma_x$. Then, it follows from (2.3) that

$$\sigma_x = \frac{1}{a}u_{xx} - \frac{h}{a}E_x = \frac{1}{a}u_{xx} + \frac{h^2}{a\beta}\sigma_x.$$

Thus,

$$\left(1 - \frac{h^2}{a\beta}\right)\sigma_x = \frac{1}{a}u_{xx}.$$

Let $\gamma = (1 - h^2/a\beta)^{-1}$, then

$$\sigma_x = \frac{\gamma}{a}u_{xx}. \tag{2.4}$$

Inserting (2.4) into (1.5) yields

$$u_{tt} - \frac{\gamma}{a\varphi}u_{xx} = 0, \tag{2.5}$$

which is the wave equation for the horizontal displacement u, with wave velocity $c^2 = \gamma/a\rho$. Note that c depends on the piezoelectric effect via γ which depends on β and h.

Initially we may prescribe

$$u = u_0 \text{ and } u_t = v_0. \tag{2.6}$$

The relevant boundary conditions are

$$u = 0 \text{ at } x = 0. \tag{2.7}$$

since the rod is fixed there.

The boundary condition at the other end is more interesting. We assume that the potential is $\varphi(0,t) = 0$ and $\varphi(L,t) = \overline{\varphi}(t)$ is prescribed function of t. Then,

$$E(L,t) = \frac{\partial\varphi}{\partial x}(L,t). \tag{2.8}$$

Since the edge $x = L$ is free, there is no stress there and $\sigma(L,t) = 0$, hence

$$u_x(L,t) = hE(L,t) = -h\varphi_x(L,t). \tag{2.9}$$