# APM 542, Winter 2006
# Solution of the Final Exam,

April 24

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You have 3 hours and you have to answer 12 questions. Answer 9 out of questions 1–12, and you have to answer questions 13 – 15. Mark clearly which three questions are **not** to be graded. Each one of the first 9 questions is worth 18 points and the last three 12 points each (total of 200). You may use two sheets of paper freely written on four sides. Please attach them to the exam. Show full logic for full credit.

1. Find a general solution of the the initial value problem, and describe the type and stability of the critical point,

$$y_1' = 2y_1 + 4y_2, \qquad y_2' = 3y_1 + y_2.$$

A: We write the system as

$$\mathbf{y}' = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A\mathbf{y}.$$

Then, $\det A = -10 = q$ and the origin $(0,0)$ is a saddle point. Next

$$0 = \det A = \begin{vmatrix} 2-\lambda & 4 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10.$$

Therefore, $\lambda_1 = 5$ and $\lambda_2 = -2$. The associated eigenvectors are $\mathbf{v}_1^T = (4,3)^T$ and $\mathbf{v}_2^T = (1,-1)$, respectively.

Finally, we obtain that the general solution is

$$\mathbf{y} = c_1\mathbf{v}_1 e^{5t} + c_2\mathbf{v}_2 e^{-2t},$$

where $c_1$ and $c_2$ are constants determined by the initial conditions.

2. Use the Cauchy-Riemann equations to find where the function $f(z) = Re(z^2) + z + 1$ is analytic.

A: We write $f = u + iv$, and then

$$u = x^2 - y^2 + x + 1, \qquad v = y.$$

Thus, $u_x = 2x + 1,\ u_y = -2y$ while $v_y = 1,\ v_x = 0$. The Cauchy-Riemann equations hold only at the origin, and we conclude that the function is not analytic at any point since a function cannot be analytic at only one point.

3. The propagation of sound in a very long pipe is governed by the equation

$$u_{tt} - 25u_{xx} = 0,$$

where $u$ is the pressure. Initially the pressure and its rate are

$$u(x,0) = \begin{cases} 1 & \text{if } -1 \le x \le 1, \\ 0 & \text{otherwise,} \end{cases} \qquad u_t(x,0) = \begin{cases} 1 & \text{if } -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

What will be the sound pressure at $x = 10$ at times $t = 1,\ t = 2, t = 10$ $sec$?

A: Using D'Alembert's solution with $c = 5,\ f(x) = u(x,0)$ and $g(x) = u_t(x,0)$, we obtain

$$u(x,t) = \frac{1}{2}(f(x+5t) + f(x-5t)) + \frac{1}{10}\int_{x-5t}^{x+5t} g(s)\,ds.$$

Therefore,

$$u(10,1) = \frac{1}{2}(f(10+5) + f(10-5)) + \frac{1}{10}\int_{10-5}^{10+5} g(s)\,ds = \frac{1}{10}\int_{5}^{15} g(s)\,ds = 0.$$

$$u(10,2) = \frac{1}{2}(f(20) + f(0)) + \frac{1}{10}\int_{0}^{20} g(s)\,ds = \frac{1}{2} + \frac{1}{10}\int_{0}^{1} ds = \frac{1}{2} + \frac{1}{10} = 0.6.$$

$$u(10,10) = \frac{1}{2}(f(60) + f(-40)) + \frac{1}{10}\int_{-40}^{60} g(s)\,ds = \frac{1}{10}\int_{-1}^{1} ds = \frac{2}{10} = 0.2.$$

4. Given the initial temperature distribution on the infinite interval $-\infty < x < \infty$,

$$u(x,0) = \begin{cases} 1 & \text{if } \; -1 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

find an integral expression for the temperature $u(x,t)$.

A: Using the integral formula for the solution with $f(x) = u(x,0)$ we obtain

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi) \exp\left(-\frac{(x-\xi)^2}{4c^2 t}\right) d\xi.$$

Therefore, with the $f(\xi) = u(\xi, 0)$ above, the solution is

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-1}^{1} \exp\left(-\frac{(x-\xi)^2}{4c^2 t}\right) d\xi.$$

5. Find the radius of convergence of the series

$$\sum_{n=3}^{\infty} 3n^2(n-1)2^n z^{3n}.$$

A: We have

$$R^3 = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{3n^2(n-1)2^n}{3(n+1)^2 n 2^{n+1}} = \frac{1}{2}.$$

Thus, $R = 2^{-1/3}$.

6. Determine the stability of all the critical points of the equation

$$y'' - \mu(1-y^2)y' + y = 0.$$

Describe the nonlinearity.

A: We write the equation as a first order system. Let $y_1 = y$ and $y_2 = y'$, then

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -y_1 + \mu(1-y_1^2)y_2. \end{aligned}$$

The critical points are the zeros of the right-hand sides, and $(0,0)$ is the only one.

Linearization about $(0,0)$ yields the system

$$y_1' = y_2,$$
$$y_2' = -y_1 + \mu y_2.$$

The matrix is

$$\begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$$

and then $p = \mu$, $q = 1$, $\Delta = p^2 - 4q = \mu^2 - 4$ and we conclude that when $\mu > 0$ the point is unstable, if $\mu > 2$ it is node out, if $0 < \mu < 2$ it is spiral out. When $\mu < 0$ the point is stable, if $|\mu| > 2$ it is node in, if $0 < |\mu| < 2$ it is spiral in.

7. Find the line integral counterclockwise, when $C$ is the unit circle,

$$\oint_C \frac{e^z}{ze^z - 5zi}\, dz$$

A: We note that the equation $e^z = 5i$ has the solutions

$$z_n = \ln 5 + \frac{\pi i}{2} \pm 2n\pi i, \qquad n = 0, 1, 2, \ldots$$

and so it has no zeros in the unit circle. Thus, it follows from the Cauchy Integral Formula that

$$\oint_C \frac{e^z}{ze^z - 5zi}\, dz = \oint_C \frac{1}{z}\frac{e^z}{e^z - 5i}\, dz = 2\pi i \left( \frac{e^z}{e^z - 5i} \right)_{z=0} = \frac{2\pi i}{1 - 5i}.$$

8. Integrate the function $f(z) = \overline{z} + z^2$ counterclockwise around the triangle with vertices $0, 1, 1 + i$.

A: We note that the part $z^2$ is analytic in $\mathbb{C}$ and so does not contribute to the integral in view of the Cauchy Theorem.

So we need to integrate $f(z) = \overline{z}$. We let $C = C_1 + C_2 + C_3$, where the three segments are parametrized by

$$C_1 : z(t) = t, \qquad C_2 : z(t) = 1+ti, \qquad C_3 : z(t) = (1+i)(1-t), \qquad 0 \le t \le 1.$$

Then on $C_1$ : $\overline{z}(t) = t$ and $z' = 1$; on $C_2$ : $\overline{z}(t) = 1 - ti$ and $z' = i$; on $C_3$ : $\overline{z}(t) = (1-i)(1-t)$ and $z' = -(1+i)$. Thus,

$$\oint_C \overline{z}\, dz = \int_{C_1} \overline{z}(t)\, dz + \int_{C_2} \overline{z}(t)\, dz + \int_{C_3} \overline{z}(t)\, dz$$

$$= \int_0^1 t\, dt + \int_0^1 (1-ti)i\, dt + \int_0^1 (1-i)(1-t)(-1)(1+i)\, dt$$

$$= \frac{1}{2}t^2|_0^1 + (it + \frac{1}{2}t^2)|_0^1 - 2(t - \frac{1}{2}t^2)|_0^1 = i.$$

9. Find the integral of the function $f$ over a closed curve $C$ that encloses the points $z = 0,\ z = 1$, where

$$f(z) = \frac{\sin(z)}{z(1-z)}.$$

A: We use the Residue Theorem, thus

$$\oint_C \frac{\sin(z)}{z(1-z)}\, dz = 2\pi i(\text{Res}_{z=0} f(z) + \text{Res}_{z=1} f(z))$$

So we compute the residues, noting that the function has simple poles at $z = 0$ and $z = 1$

$$\text{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{z\sin(z)}{z(1-z)} = \lim_{z\to 0} \frac{\sin(z)}{(1-z)} = \frac{0}{1} = 0,$$

(so $z = 0$ is NOT a pole!) and

$$\text{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{(z-1)\sin(z)}{z(1-z)} = -\lim_{z\to 1} \frac{\sin(z)}{z} = \frac{-\sin 1}{1} = -\sin 1.$$

We conclude that

$$\oint_C \frac{\sin(z)}{z(1-z)}\, dz = -2\pi i \sin 1.$$

10. Determine the location and type of all critical points of the system

$$y_1' = y_1 + y_2, \qquad y_2' = y_1 - y_1^3.$$

A: We note that the critical points are

$$y_1 + y_2 = 0, \qquad y_1(1 - y_1^2) = 0.$$

Then $y_2 = -y_1$ and $y_1 = 0, \pm 1$, and so the critical points are

$$P_1 = (0,0), \qquad P_2 = (1,-1), \qquad P_2 = (-1,1).$$

Linearization at $P_1$ yields

$$\begin{aligned} y_1' &= y_1 + y_2, \\ y_2' &= y_1. \end{aligned}$$

The matrix is

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

so $p = 1$, $q = -1$ and the origin $(0,0)$ is a saddle point- so it is unstable.

To linearize at $P_2$ we set $\overline{y_1} = y_1 - 1$ and $\overline{y_2} = y_1 + 1$. The system is transformed into

$$\begin{aligned} \overline{y_1}' &= (\overline{y_1} + 1) + (\overline{y_2} - 1) = \overline{y_1} + \overline{y_2}, \\ y_2' &= (\overline{y_1} + 1)(1 - (\overline{y_1} + 1)^2) = (\overline{y_1} + 1)(\overline{y_1}^2 - 2\overline{y_1}) \approx -2\overline{y_1}. \end{aligned}$$

where we assumed that the new variables are close to zero. The matrix is

$$\begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix},$$

so $p = 1$, $q = 2$ and $\Delta = -7 < 0$ and the point $(1,-1)$ is an unstable spiral out. A similar construction yields that the other critical point $(-1,1)$ is an unstable spiral out, too.

11. Where does the sum converge

$$f(z) = \sum_{n=1}^{\infty} n z^{n-1}.$$

Where is it analytic? What is $f(z)$?

A: We compute the radius of convergence

$$R = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

The sum converges inside the unit circle and there it defines an analytic function. Next, we note that

$$f(z) = \sum_{n=1}^{\infty} n z^{n-1} = \sum_{n=1}^{\infty} (z^n)' = \left(\sum_{n=1}^{\infty} z^n\right)' = \left(\sum_{n=0}^{\infty} z^n\right)'$$
$$= \left(\frac{1}{1-z}\right)' = \frac{1}{(1-z)^2}, \qquad |z| < 1.$$

We note that the function $f(z)$ given by the series is defined only for $|z| < 1$, and there it is analytic.

12. Find the Laurent series of the function $f(z) = \sin z/z^3$, and find $R$ such that the series converges for $0 < |z| < R$.

A: The Taylor series of $\sin z$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 + \dots$$

with radius of convergence $R = \infty$. Then

$$\frac{\sin z}{z^3} = \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n-2}}{(2n+1)!} = \frac{1}{z^2} - \frac{1}{6} + \frac{1}{120}z^2 + \dots$$

The series converges for $0 < |z| < \infty$.

13. (You have to answer this question) Determine the types of all the singularities of the function

$$f(z) = \frac{\sin^2 z}{z^2} + \frac{2}{z-i}.$$

A: The second term on the right-hand side is a simple pole at $z = i$. So we need only to find out the singularities of the first term, and clearly the only candidate is $z = 0$. However, recalling that

$$\lim_{z\to 0} \frac{\sin z}{z} = 1,$$

we find that the first term does NOT have a singularity. Another way to obtain this is to use the series expansion of $\sin z$ (Question 12) and note that the first term in the product is $z^2$ and all the other terms have higher powers of $z$.

We conclude that $z = i$ is the only singularity and it is a simple pole.

14. (You have to answer this question) Use separation of variables to find the solutions $u(x, y)$ of

$$u_x + u_y = (x + y)u.$$

A: Let $u(x, y) = F(x)G(y)$. Then the equation is

$$F'G + FG' = (x + y)FG.$$

Here the prime denotes the derivative with respect to the argument ($F' = dF/dx$, $G' = dG/dy$). Dividing by $FG$ ( assumed not to be identically zero) and rearranging yields

$$\frac{F'}{F} - x = y - \frac{G'}{G} = k.$$

Hence we obtain two ordinary differential equations

$$F' = (x + k)F, \qquad G' = (y - k)G.$$

The solutions are

$$F(x) = a\exp(\frac{1}{2}x^2 + kx), \qquad G(y) = b\exp(\frac{1}{2}y^2 - ky),$$

where $a$ and $b$ are two arbitrary constants. Therefore,

$$u(x, y) = F(x)G(y) = c\exp(\frac{1}{2}(x^2 + y^2) + k(x - y)).$$

15. (You have to answer this question) Find the residues at the singularities of the function

$$f(z) = \frac{1}{1 - e^z}.$$

A: The order of a pole of $f$ at $z = z_0$ is the same as the order of the zero of $1 - e^z$ at $z_0$. Therefore, we find the zeros of the denominator,

$$e^z = 1 \quad \Longrightarrow \quad z = \pm 2\pi n i, \quad n = 0, 1, 2, \ldots.$$

Each of these zeros is simple and so the poles of $f$ are simple. Then the residue formula ((4) on page 783) yields

$$\text{Res}_{z=0}\frac{1}{1-e^z} = \frac{1}{(1-e^z)'}|_{z=0} = \frac{1}{-e^z}|_{z=0} = -1.$$

The residues at all the other singular points are the same.