

APM 542, Winter 2004
Final EXAM – Solutions; April 26

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You have 3 hours and you have to answer 12 questions. Answer 9 out of questions 1–12, and you have to answer questions 13 – 15. Mark clearly which three questions are **not** to be graded. Each question is worth 16.5 points (total of 200). You may use two sheets of paper freely written on four sides. Please attach them to the exam. Show full logic for full credit.

Good luck!

1. Find the eigenvalues of the initial value problem, and describe the critical point,

$$y_1' = y_1 + y_2, \quad y_2' = 4y_1 + y_2.$$

A: We have

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Then, the critical point is $(0, 0)$ and since

$$q = \det A = 1 - 4 = -3 < 0,$$

the point is a saddle point. The eigenvalues are found from

$$|A - \lambda I| = \begin{vmatrix} 1 - \lambda & 1 \\ 4 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^2 - 4 = 0,$$

thus, $\lambda_1 = -1$ and $\lambda_2 = 3$.

2. Use the Cauchy-Riemann equations to find where the function

$$f(z) = \frac{1}{z - i}$$

is analytic.

A: We have

$$f(z) = \frac{1}{z - i} = \frac{\bar{z} + i}{(z - i)(\bar{z} + i)} = \frac{x - i(y - 1)}{x^2 + (y - 1)^2}.$$

Thus

$$u = \frac{x}{x^2 + (y - 1)^2} \quad v = -\frac{y - 1}{x^2 + (y - 1)^2}.$$

A straightforward computation yields

$$\begin{aligned} u_x &= \frac{(y - 1)^2 - x^2}{(x^2 + (y - 1)^2)^2}, & u_y &= \frac{2x(y - 1)}{(x^2 + (y - 1)^2)^2}, \\ v_x &= \frac{-2x(y - 1)}{(x^2 + (y - 1)^2)^2}, & v_y &= \frac{(y - 1)^2 - x^2}{(x^2 + (y - 1)^2)^2}. \end{aligned}$$

We conclude that $u_x = v_y$ and $u_y = -v_x$ everywhere except for $z = i$, i.e., $(0, 1)$. So the function is analytic everywhere except at this point.

3. The propagation of sound in a long pipe is governed by the equation

$$u_{tt} - 4u_{xx} = 0,$$

where u is the pressure. If a short burst of noise of constant pressure $f = 2$ takes place initially for $-2 < x < 2$, ($g = 0$) what will be the sound pressure in $x = 6$ at times $t = 1.9$, $t = 2.1$, $t = 6$ sec?

A: D'Alembert's solution is

$$u(x, t) = \frac{1}{2}(f(x + 2t) + f(x - 2t)),$$

since $c = 2$. Therefore,

$$u(6, 1.9) = \frac{1}{2}(f(6 + 3.8) + f(6 - 3.8)) = \frac{1}{2}(f(9.8) + f(2.2)) = 0 + 0 = 0;$$

Similarly

$$u(6, 2.1) = \frac{1}{2}(f(10.2) + f(1.8)) = 1; \quad u(6, 6) = \frac{1}{2}(f(18) + f(-12)) = 0.$$

4. Let $G = G(u)$ be the potential energy function

$$G(u) = \int_{\Omega} \|\nabla u\|^2 dA - 2 \int_{\Omega} f u dA,$$

of a membrane occupying a domain Ω in the plane, with boundary Γ , and let

$$K = \{v : \phi \leq v, \quad v = g \text{ on } \Gamma\},$$

where $z = \phi(x, y)$ is the obstacle under the membrane and $z = g$ is the shape of the rim.

Describe the classical equations and conditions that are satisfied by the minimizer $u \in K$ of G ,

$$G(u) \leq G(v), \quad \text{for all } v \in K.$$

A: The problem is that of a stretched membrane lying above a rigid obstacle. Classically, the problem can be written as

$$\begin{aligned} \nabla^2 u + f &\leq 0 && \text{in } \Omega, \\ \phi &\leq u && \text{in } \Omega, \\ (\nabla^2 u + f)(u - \phi) &= 0 && \text{in } \Omega, \\ u &= g && \text{on } \Gamma. \end{aligned}$$

5. Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{i^n n^3}{3^n} z^{2n}.$$

A: We have that the radius of convergence is

$$R^2 = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{n^3 3^{n+1}}{3^n (n+1)^3} = \lim_{n \rightarrow \infty} 3 \left(\frac{n}{n+1} \right)^3 = 3.$$

We conclude that the radius of convergence is

$$R = \sqrt{3}.$$

6. Compute the integral over $C : |z| = \pi$,

$$I = \oint_C \frac{e^z + 3 + z}{z(1-z)} dz.$$

A: The function has two simple poles at $z = 0$ and $z = 1$. The residues are:

$$\operatorname{Res}_{z \rightarrow 0} \frac{e^z + 3 + z}{z(1-z)} = \lim_{z \rightarrow 0} \frac{z(e^z + 3 + z)}{z(1-z)} = \lim_{z \rightarrow 0} \frac{e^z + 3 + z}{1-z} = 4.$$

$$\operatorname{Res}_{z \rightarrow 1} \frac{e^z + 3 + z}{z(1-z)} = \lim_{z \rightarrow 1} \frac{(z-1)(e^z + 3 + z)}{z(1-z)} = -\lim_{z \rightarrow 1} \frac{e^z + 3 + z}{z} = -e - 4.$$

Using residue integration we conclude that

$$I = 2\pi i(4 - e - 4) = -2\pi e i.$$

7. Find the phase-plane trajectories of the equation

$$yy'' - (y')^2 = 0.$$

A: Let $y_1 = y$ and $y_2 = y'$. Then, the equation can be written as

$$y_1 \frac{dy_2}{dt} = y_2 \frac{dy_1}{dt} \implies \frac{1}{y_2} \frac{dy_2}{dt} = \frac{1}{y_1} \frac{dy_1}{dt}.$$

Therefore,

$$(\ln y_2)' = (\ln y_1)',$$

and then $\ln y_2 = \ln y_1 + c$, so

$$\ln \left(\frac{y_2}{y_1} \right) = c.$$

We conclude that $y_2/y_1 = c$, or $y_2 = cy_1$, and the trajectories lie on the rays through the origin.

8. Integrate the function $f(z) = \operatorname{Re}(z^2)$ counterclockwise around the triangle with vertices $0, 1, i$.

A: We can represent the path of integration as $C = C_1 + C_2 + C_3$ where

$$C_1 : z(t) = t, \quad C_2 : z(t) = (1 - t) + it, \quad C_3 : z(t) = (1 - t)i,$$

for $0 \leq t \leq 1$. Next, $\operatorname{Re}(z^2) = x^2 - y^2$, thus

$$\begin{aligned} \oint_C \operatorname{Re}(z^2) dz &= \int_{C_1} \operatorname{Re}(z^2) dz + \int_{C_2} \operatorname{Re}(z^2) dz + \int_{C_3} \operatorname{Re}(z^2) dz \\ &= \int_0^1 (t^2) dt + (-1 + i) \int_0^1 (1 - 2t) dt + i \int_0^1 (1 - t)^2 dt \\ &= \frac{1}{3} t^3 \Big|_0^1 + (-1 + i)(t - t^2) \Big|_0^1 - \frac{i}{3} (1 - t)^3 \Big|_0^1 = \frac{1}{3} + \frac{i}{3} = \frac{1}{3}(1 + i). \end{aligned}$$

9. Find the integral of the function f over closed curve C that encloses the points $z = 0$, $z = -1$, where

$$f(z) = \frac{2z - 1}{z^2 + z}.$$

A: We note that

$$f(z) = \frac{2z - 1}{z^2 + z} = \frac{2z - 1}{z(z + 1)}.$$

Then the integral can be evaluated using the residue theorem,

$$I = \oint_C f(z) dz = 2\pi i (\operatorname{Res}_{z \rightarrow 0} f(z) + \operatorname{Res}_{z \rightarrow -1} f(z)).$$

Next,

$$\operatorname{Res}_{z \rightarrow 0} f(z) = \lim_{z \rightarrow 0} \frac{z(2z - 1)}{z(z + 1)} = -1, \quad \operatorname{Res}_{z \rightarrow -1} f(z) = 3.$$

We conclude that $I = 4\pi i$.

10. Determine the location and type of all critical points of the system

$$y_1' = y_2, \quad y_2' = y_1 - y_1^3.$$

A: The critical points are the zeros of the right hand sides; thus, these are $(0, 0), (\pm 1, 0)$. We begin with the type of $(0, 0)$, and neglecting y_1^3 , by linearizing, we find

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Then, $q = \det A = -1 < 0$, and the point is a saddle point. Next, we linearize about $(1, 0)$ by introducing $\bar{y}_1 = y_1 - 1$ and $\bar{y}_2 = y_2$. Then, $y_1 - y_1^3 = (\bar{y}_1 + 1) - (\bar{y}_1 + 1)^3 = -\bar{y}_1^3 - 3\bar{y}_1^2 - 2\bar{y}_1$. Neglecting the terms with \bar{y}_1^3 and \bar{y}_1^2 we find

$$\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}.$$

Then, $q = \det A = 2 > 0; p = 0$, and the point is a center, and similarly for the critical point $(-1, 0)$. The trajectories are closed ellipses.

11. Find the sum

$$S = \sum_{n=0}^{\infty} (n+2)(n+1) z^n.$$

A: The easiest way to obtain the solution is to note that

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \implies \quad f(z)'' = \frac{2}{(1-z)^3} = \sum_{n=0}^{\infty} (n+2)(n+1) z^n.$$

Therefore,

$$S = \frac{2}{(1-z)^3},$$

and the equality holds where the series converges, that is $|z| < 1$.

12. A beam is simply supported at both ends, $x = 0$ and $x = L$, so the boundary conditions are

$$u(0, t) = 0, \quad u_{xx}(0, t) = 0, \quad u(L, t) = 0, \quad u_{xx}(L, t) = 0.$$

If we set $u(x, t) = F(x)G(t)$, what is the equation for the natural frequencies?

A: We have $\ddot{G}F - c^2GF'''' = 0$, thus

$$\frac{F''''}{F} = -\beta^4, \quad \ddot{G} + \lambda^2G = 0.$$

where $\lambda = c\beta^2$ is the frequency. From the boundary conditions we have $F(0) = F''(0) = 0$ and $F(L) = F''(L) = 0$. The solution to the spatial problem is

$$F(x) = A \cos(\beta x) + B \cosh(\beta x) + C \sin(\beta x) + D \sinh(\beta x),$$

and

$$F''(x) = -\beta^2 A \cos(\beta x) + \beta^2 B \cosh(\beta x) - \beta^2 C \sin(\beta x) + \beta^2 D \sinh(\beta x).$$

It follows now from the boundary conditions $F(0) = F''(0) = 0$ that $A = B = 0$. Then, the boundary conditions $F(L) = F''(L) = 0$ yield

$$C \sin(\beta L) + D \sinh(\beta L) = 0, \quad -C \sin(\beta L) + D \sinh(\beta L) = 0.$$

Thus, $C = 0$ and the equation for the β s is

$$\sin(\beta L) = 0, \quad \implies \quad \beta_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots$$

We conclude that the frequencies of vibrations are

$$\lambda_n = \frac{cn^2\pi^2}{L^2}, \quad n = 1, 2, 3, \dots$$

13. (You have to answer this question) Find the Taylor series, at $z = 0$, of the function

$$f(z) = \int_0^z \frac{\sin t}{t} dt.$$

A: The series expansion for the $\sin(t)$ function is

$$\sin t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!},$$

which converges everywhere in the complex plane \mathbb{C} , and so

$$\frac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}.$$

We now integrate this expression term by term, which is allowed since the series converges, thus,

$$\begin{aligned} \int_0^z \frac{\sin t}{t} dt &= \int_0^z \left(\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!} \right) dt \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \int_0^z t^{2n} dt \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n)!(2n+1)^2}. \end{aligned}$$

14. (You have to answer this question) The vibrations of a string that is attached $0 \leq x \leq L$ to the wall at $x = 0$ and satisfies the third boundary condition at $x = L$ are described by

$$u_{tt} - c^2 u_{xx} = 0,$$

$$u(0, t) = 0, \quad u_x(L, t) + \alpha u(L, t) = 0,$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = 0.$$

Find the equation for the frequencies (in time) of the vibrations.

A: We use separation of variables and let $u(x, t) = F(x)G(t)$. Then, $\ddot{G}F - c^2GF'' = 0$ and so

$$\frac{F''}{F} = -\beta^2, \quad \ddot{G} + \lambda^2G = 0.$$

where $\lambda = c\beta$ is the frequency, and $\beta \neq 0$. From the boundary conditions we have $F(0) = 0$ and $F'(L) + \alpha F(L) = 0$. The solution to the spatial problem is

$$F(x) = A \cos(\beta x) + B \sin(\beta x),$$

and

$$F'(x) = -\beta A \sin(\beta x) + \beta B \cos(\beta x).$$

It follows now from the boundary condition $F(0) = 0$ that $A = 0$. Then, the boundary condition $F'(L) + \alpha F(L) = 0$ yields

$$\beta \cos(\beta L) + \alpha \sin(\beta L) = 0.$$

Where $B \neq 0$ and canceled out and the equation for the β s is

$$\beta = -\alpha \tan(\beta L),$$

which has an infinite number of solutions $\beta_n, \quad n = 1, 2, 3, \dots$

We conclude that the frequencies of vibrations are

$$\lambda_n = c\beta_n, \quad n = 1, 2, 3, \dots$$

15. (You have to answer this question) Find the Laurent series and the radius of convergence R , that converges at $z_0 = i$, for $0 < |z - i| < R$, of

$$f(z) = \frac{1}{z^2 + 1}.$$

A: We write

$$\frac{1}{z^2 + 1} = \frac{1}{(z + i)(z - i)} = \frac{1}{(z - i)^2 \left(1 + \frac{2i}{z - i}\right)}.$$

Now,

$$\frac{1}{\left(1 + \frac{2i}{z - i}\right)} = \sum_{n=0}^{\infty} \frac{(-2i)^n}{(z - i)^n},$$

so we conclude that

$$f(z) = \sum_{n=0}^{\infty} \frac{(-2i)^n}{(z-i)^{n+2}}.$$

It is straightforward to see that

$$R = \left| \frac{(-2i)^n}{(-2i)^{n+1}} \right| = \frac{1}{2}.$$

Another expansion is possible, and is obtained as follows. We write

$$\frac{1}{(z-i)(z+i)} = \frac{1}{(z-i)(z-i+2i)} = \frac{1}{2i(z-i)} \frac{1}{1 + \frac{z-i}{2i}}.$$

Now,

$$\frac{1}{1 + \frac{z-i}{2i}} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(2i)^{n+1}}.$$

Noting that $(-1)^n/i^n = i^n$ we obtain, after multiplying by the term $1/2i(z-i)$ and rearranging,

$$f(z) = \frac{1}{z^2 + 1} = \sum_{n=0}^{\infty} (i)^{n-1} \frac{(z-i)^n}{2^{n+1}}.$$

The radius of convergence is $R = 2$.