APM 542, Winter 2006
EXAM - 2, March 27

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You have 100 minutes to answer 8 questions. Answer 6 out of questions 1–8, each one is worth 14 points, and you have to answer questions 9 and 10, which are worth 10 points each (to total of 100 points). Mark clearly which two questions are **not** to be graded. Show full logic for full credit. You may use one page written freely on both sides. **Good luck!**

1. Evaluate the complex line integral

$$\int_C \overline{z}\, dz,$$

where C is the path $x = 2t$, $y = t^2$, for $-2 \leq t \leq 2$.

A: We parametrize the path C as $z(t) = 2t + it^2$ for $-2 \leq t \leq 2$, and so $z'(t) = 2 + 2it$. Next, on the path $\overline{z} = 2t - it^2$. Therefore,

$$\int_C \overline{z}\, dz = \int_{-2}^{2} (2t - it^2)(2 + 2it)\, dt = \int_{-2}^{2} (4t + 2t^3 + 2it^2)\, dt$$

$$= (2t^2 + \frac{1}{2}t^4 + \frac{2}{3}it^3)|_{-2}^{2} = 16 + \frac{16}{3}i - (16 - \frac{16}{3}i) = \frac{32}{3}i.$$

2. Assume that the function $f(z)$ is analytic in a bounded domain D, and $|f(z)| = m$ in D, where m is a real constant. What can you say about f?

A: Then $f(z) = const$. Indeed, if $m = 0$ then $f = 0$. If $m \neq 0$ we write $f = u + iv$, then $|f|^2 = u^2 + v^2 = m^2$. We differentiate with respect to x and y, divide by 2, and obtain

$$uu_x + vv_x = 0, \qquad uu_y + vv_y = 0.$$

Next, we use the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$, multiply the first equation by u, the second by v, and find

$$u^2 u_x - vuu_y = 0, \quad uvu_y + v^2 u_x = 0.$$

Adding both equations yields

$$0 = (u^2 + v^2)u_x = m^2 u_x = 0 \implies u_x = 0.$$

Similarly, multiplying the first equation by v and the second by u leads to

$$uvu_x + v^2 v_x = uvv_y - v^2 u_y = 0, \qquad u^2 u_y + uvv_y = 0.$$

Subtracting the equations yields

$$0 = (u^2 + v^2)u_y = m^2 u_y = 0 \implies u_y = 0.$$

Since $u_x = 0$ and $u_y = 0$ it follows that $u = const$, and then it follows from the Cauchy-Riemann equations that $v = const.$, and so $f = const.$

3. Let z be a complex number, show that

$$\cos^2 z + \sin^2 z = 1.$$

A: We have

$$\cos^2 z = \frac{1}{4}(e^{iz} + e^{-iz})^2 \qquad \sin^2 z = -\frac{1}{4}(e^{iz} - e^{-iz})^2.$$

Therefore,

$$\cos^2 z + \sin^2 z = \frac{1}{4}(e^{2iz} + 2 + e^{2-iz} - e^{2iz} + 2 - e^{2-iz}) = 1.$$

4. Find the complex line integral $\oint_C f(z)\,dz$ of the function

$$f(z) = z^2 + \frac{1}{z} + \frac{1}{z-4} + \frac{1}{(z-4)^2},$$

where C is either (i) the circle $|z| = 1$, or (ii) the circle $|z| = 12$. (You do not need to compute the integrals, but you must explain carefully which formulas you used, and fully justify your answer).

A: Since z^2 is analytic everywhere in $\mathbb{C}$, its contribution to either line integral is zero. $1/z$ has a singularity at the origin, and $\oint_C (1/z)\,dz = 2\pi i$ for either curve C. The functions $1/(z-4)$ and $(z-4)^2$ are analytic on and inside the first curve. For the second curve we have

$$\oint_{|z|=12} \frac{1}{z-4}\,dz = 2\pi i, \qquad \oint_{|z|=12} \frac{1}{(z-4)^2}\,dz = 0.$$

Therefore,

$$\oint_{|z|=1} f(z)\,dz = 2\pi i, \qquad \oint_{|z|=12} f(z)\,dz = 4\pi i.$$

5. Determine where do the following functions satisfy the Cauchy-Riemann equations.

$$(i) \qquad f(z) = iz + |z^2|, \qquad (ii) \quad f(z) = z^2 - iz, \qquad (iii) \quad f(z) = \ln(1/z).$$

A: (i) We have $f = u + iv = (x^2 + y^2 - y) + ix$, and so $v_y = 0$ while $u_x = 2x$, also $u_y = 2y - 1$ and $v_x = 1$, and the Cauchy-Riemann equations are satisfied only at $z = 0$.
(ii) $f' = 2z - i$, so f is analytic in $\mathbb{C}$ and the Cauchy-Riemann equations are satisfied everywhere.
(iii) We have $\ln(1/z) = \ln 1 - \ln(z) \pm 2n\pi i = -\ln(z) \pm 2n\pi i$. Then,

$$f' = -\frac{1}{z}.$$

The function is analytic everywhere except at $z = 0$, and so it satisfies the Cauchy-Riemann equations everywhere except at $z = 0$.

6. Evaluate the complex line integral

$$\int_C \frac{2}{z}\, dz,$$

where C is the straight line segment from $z = 1 + i$ to $z = 9 + 9i$.

A: We have that $f = 2/z = (2\ln(z))' = F'$. Since $F = 2\ln(z)$ is analytic in any neighborhood of C, we obtain

$$\int_C \frac{2}{z}\, dz = F(9+9i) - F(1+i) = 2\ln\left(\frac{9+9i}{1+i}\right) = 2\ln 9.$$

7. Find all z such that $e^z = 1 + 2i$.

A: Taking the ln of both sides yields

$$z = \ln(1+2i) = \ln(\sqrt{5}) + i\text{Arg}(1+2i) \pm 2\pi n i,$$

for $n = 0, 1, 2, \ldots$. Finally,

$$\text{Arg}(1+2i) = \tan^{-1}(2).$$

8. Write the following expressions as $a + ib$:
 (i) $\sin^2(1+i)$; (ii) $\sin(e^i)$.

A: (i) We can use either of two ways. We have $\sin(z) = (\exp(iz) - \exp(-iz))/2i$, thus,

$$\sin^2(1+i) = -\frac{1}{4}\left(e^{2i(1+i)} - 2 + e^{-2i(1+i)}\right) =$$

$$= -\frac{1}{4}\left(e^{-2+2i} - 2 + e^{2-2i}\right) = \frac{1}{4}(2 - e^{-2}(\cos 2 + i\sin 2) - e^2(\cos 2 - i\sin 2))$$

$$= \frac{1}{2}(1 - \cos 2\cosh 2) + \frac{1}{2} i \sin 2 \sinh 2.$$

(ii) We have $e^i = \cos(1) + i\sin(1)$, thus,

$$\sin(e^i) = \sin(\cos(1) + i\sin(1)) = \sin(\cos(1))\cosh(\sin(1)) + i\cos(\cos(1))\sinh(\sin(1)).$$

9. (You have to answer this question) Evaluate the line integral

$$I = \oint_C \left(\frac{e^z}{(z+1)^2} + 2\overline{z} \right) dz,$$

where C is the unit circle $|z| = 2$.

A: We split the integral into two parts. Let $z = 2e^{it}$ for $0 \le t \le 2\pi$, and then $z' = 2ie^{it}$, and so

$$I_1 = 2 \oint_C \overline{z}\, dz = 2 \int_0^{2\pi} 2e^{-it} 2ie^{it}\, dt = 8i \int_0^{2\pi} dt = 16\pi i.$$

Next, we use Cauchy's Integral Formula for the derivative and obtain

$$I_2 = \oint_C \frac{e^z}{(z+1)^2}\, dz = 2\pi i (e^z)'|_{z=-1} = 2\pi e^{-1} i.$$

We conclude that

$$I = I_1 + I_2 = 16\pi i + 2\pi e^{-1} i = 2\pi i(8 + e^{-1}).$$

10. (You have to answer this question) Let z and w be two nonzero complex numbers. What is the relationship between $\ln(z/w)$ and $\ln(z)$, $\ln(w)$?

A: From the definitions

$$\ln\left(\frac{z}{w}\right) = \ln\left|\frac{z}{w}\right| + i \arg\left(\frac{z}{w}\right) \pm 2\pi n i$$

$$= \ln|z| - \ln|w| + i\mathrm{Arg}(z) - i\mathrm{Arg}(w) \pm 2\pi n i,$$

for $n = 0, 1, 2, \ldots$. Next,

$$\ln(z) = \ln|z| + i\mathrm{Arg}(z) \pm 2\pi k i, \qquad \ln(w) = \ln|w| + i\mathrm{Arg}(w) \pm 2\pi m i,$$

for $k, m = 0, 1, 2, \ldots$.

Therefore, every number $\widehat{z}$ such that $\widehat{z} = \ln(z/w)$ is also such that $\widehat{z} = \ln(z) - \ln(w)$, so the sets on both sides include the same numbers, so they are the same.