APM 542, Winter 2004
EXAM no. 2, - Solutions

M. Shillor

You have 105 minutes and you have to answer 8 questions. Answer 6 out of questions 1–8, and you have to answer questions 9 and 10. Mark clearly which two questions are **not** to be graded. Each question is worth 12.5 points (total of 100). Show full logic for full credit. You may use one page written freely on one side. **Good luck!**

1. Write the number $z = (1+2i)^2 + (i)^5 + (2i)^4$ as the sum of a real and imaginary parts.

A: We note that $(i)^5 = i$ and $(i)^4 = 1$, thus

$$z = 1 - 4 + 4i + i + 2^4 = 13 + 5i.$$

2. Complete the statement,

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2.$$

A: We have

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) + (z_1 - z_2)(\overline{z_1} - \overline{z_2})$$

$$= z_1\overline{z_1} + z_2\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_2} + z_1\overline{z_1} - z_2\overline{z_1} - z_1\overline{z_2} + z_2\overline{z_2}$$

$$= 2z_1\overline{z_1} + 2z_2\overline{z_2} = 2|z_1|^2 + 2|z_2|^2.$$

3. Let z be a complex number, show that

$$\cosh^2 z + \sinh^2 z = \cosh 2z.$$

A: We have

$$\cosh^2 z = \frac{1}{4}(e^z + e^{-z})^2 = \frac{1}{4}(e^{2z} + 2 + e^{-2z}),$$

and

$$\sinh^2 z = \frac{1}{4}(e^z - e^{-z})^2 = \frac{1}{4}(e^{2z} - 2 + e^{-2z}).$$

Hence

$$\cosh^2 z + \sinh^2 z = \frac{1}{2}(e^{2z} + e^{-2z}) = \cosh 2z.$$

4. Which of the following functions is analytic? If it is analytic, indicate where in $\mathbb{C}$. (Explain carefully!)

$$\begin{array}{lrcl} (i) & f(z) & = & (\overline{z})^2; \\ (ii) & f(z) & = & z\overline{z}; \\ (iii) & f(z) & = & \cos(z^{1/2}). \end{array}$$

A: We have

(i) $f(z) = (\overline{z})^2 = (x-iy)^2 = x^2 - y^2 - 2xyi$. So $u = x^2 - y^2$ and $v = -2xy$. Using the Cauchy-Rieamann equations we find

$$u_x = 2x, \qquad v_y = -2x,$$

therefore, the function is not analytic anywhere.

(ii) $f(z) = x^2 + y^2$. So $u = x^2 + y^2$ and $v = 0$. Using the Cauchy-Rieamann equations we find that $u_x = 2x$ and $v_y = 0$, therefore, the function is not analytic anywhere.

(iii) We differentiate and find

$$f'(z) = \frac{-1}{2z^{1/2}} \sin(z^{1/2}),$$

therefore, the function is analytic everywhere in $\mathbb{C}$, except at the origin.

5. Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} .$$

A: We use the ration test. Since $a_n = (n!)^2/(2n)!$ we have that

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{((n+1)!)^2(2n)!}{(n!)^2(2(n+1))!} = \frac{(n!)^2(n+1)^2(2n)!}{(n!)^2(2n)!(2n+1)(2n+2)}$$

$$= \frac{(n+1)^2}{2(2n+1)(n+1)} = \frac{(1+1/n)}{2(2+1/n)} \leq \frac{1}{2}.$$

We conclude that the series converges.

6. Find the line integral of the function $f(z) = 2z + 3/z$ over the circle of radius $R = 2.333$.

A: The first part of the function is analytic, so the integral vanishes by the Cauchy Theorem. The integral of $3/z$ over any circle containing the origin is $6\pi i$. Thus,

$$\oint_C \left(2z + \frac{3}{z}\right) dz = 6\pi i.$$

7. Find $i^{1/6}$.

A: We have

$$i = \exp(i\pi/2) = \cos(\pi/2) + i\sin(\pi/2).$$

Therefore, $\theta = \pi/2$ and

$$i^{1/6} = \cos\left(\frac{\pi/2 + 2\pi k}{6}\right) + i\sin\left(\frac{\pi/2 + 2\pi k}{6}\right), \qquad \text{for } k = 0, 1, \ldots, 5.$$

8. Find the line integral

$$\oint_C \frac{3z^2 - 1}{z(z-1+i)^2}\, dz.$$

Here, C consists of the circle $|z| = 3$ counterclockwise and $|z| = 1$ clockwise.

A: The area enclosed between the two circles contains the point $z = 1 - i$ where the integrand is not analytic. But the function

$$f(z) = (3z^2 - 1)/z = 3z - \frac{1}{z}$$

is analytic there, Thus, by the Cauchy formula

$$I = \oint_C \frac{3z^2 - 1}{z(z-1+i)^2}\, dz = 2\pi i f'|_{1-i}.$$

Now, $f' = 3 + 1/z^2$, therefore,

$$I = 2\pi i \left(3 + \frac{1}{(1-i)^2}\right) = 2\pi i \left(3 + \frac{i}{2}\right) = -\pi + 6\pi i.$$

9. (You have to answer this question) Can the function

$$v = \frac{-y}{x^2 + y^2}$$

be the imaginary part of an analytic function $f(z) = u + iv$? If so, what is f and where is f analytic?

A: We note that for $z = x + iy$,

$$f(z) = \frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

If we write $f(z) = u + iv$ then

$$u = \frac{x}{x^2 + y^2}, \qquad v = \frac{-y}{x^2 + y^2}.$$

We conclude that the real part is u, and the function is $f(z) = 1/z + c$ which is analytic everywhere, except at $z = 0$.

10. (You have to answer this question) Find the line integral

$$\int_C (z\overline{z} + 2iz)\, dz,$$

where C is the path from $z = 0$ to $z = 1 + i$ along the parabola $y = x^2$.

A: This can be done in two ways. In the first one we parametrize the path, as below, and compute the integral. In the second one we split the integral, and rewrite it as

$$I = \int_C z\overline{z}\, dz + \int_C 2iz\, dz.$$

Next, $(iz^2)' = 2iz$ so this part is analytic and path independent, hence,

$$\int_C 2iz\, dz = (iz^2)|_0^{1+i} = i(1+i)^2 = -2.$$

The part $z\overline{z} = x^2 + y^2$ is not analytic, it is path dependent. We parametrize the path as $C:\ z(t) = t + it^2$ for $0 \le t \le 1$. Then, $z' = 1 + 2it$ and

$$\int_C z\overline{z}\, dz = \int_C (x^2 + y^2)\, d(x + iy) = \int_0^1 (t^2 + t^4)(1 + 2it)\, dt$$

$$= \int_0^1 (t^2 + t^4 + 2it^3 + 2it^5)\, dt = \left(\frac{1}{3}t^3 + \frac{1}{5}t^5 + i(\frac{1}{2}t^4 + \frac{1}{3}t^6)\right)_0^1$$

$$= \left(\frac{1}{3} + \frac{1}{5} + i(\frac{1}{2} + \frac{1}{3})\right) = \frac{8}{15} + \frac{5}{6}i.$$

We conclude that

$$I = -2 + \frac{8}{15} + \frac{5}{6}i = -\frac{22}{15} + \frac{5}{6}i.$$