

APM 542, Winter 2007
Exam 2A, March 26

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Name

You have 90 minutes and you have to answer 7 questions. Answer 5 out of questions 1–8, and you have to answer questions 9 and 10. Mark clearly which two questions are **not** to be graded. Each of the question 1–8 is worth 18 points and each of 9 and 10 is worth 15 points (total of 120). Show full logic for full credit. You may use one page written freely on two sides.

Good luck!

1. Find all the solutions of the equation

$$e^z = -4.$$

A. We have

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y) = -4.$$

Then $|e^z| = e^x = 4$, and so $x = \ln 4$. Next,

$$\cos y + i \sin y = -1.$$

Therefore, $\sin y = 0$ and $\cos y = -1$, and so $y = \pi$.

We conclude, since e^z is periodic in the y variable, that all the solutions are given by

$$z = \ln 4 + \pi i + 2n\pi i, \quad n = 1, 2, 3, \dots$$

Note that $\text{Arg } z = \pi i$.

2. Use the Cauchy-Riemann equations and one of the expressions for $f'(z)$ to show that

$$f'(z) = u_x - iu_y.$$

A. We may use the formula $f' = u_x + iv_x$, and since it follows from the Cauchy-Riemann equations that $u_y = -v_x$ we obtain the answer.

3. In an infinite bar the initial temperature is $f(x) = 2$, for $0 \leq x \leq 2$, and $f(x) = 0$ otherwise. When will the temperature at $x = 100,000$ be positive?

A. The solution of the problem can be written as

$$u(x, t) = \frac{2}{2c\sqrt{\pi t}} \int_0^2 \exp\left(-\frac{(x-s)^2}{4c^2t}\right) ds.$$

Clearly, for each $t > 0$ the integral is positive, hence the temperature at each point x is instantly positive. The speed of propagation of heat in infinite

4. Let $f(z)$ be analytic and $\operatorname{Re} f(z) = \text{const.}$ Does it imply that $f(z) = \text{const.}$?

A. We write

$$f(z) = u + iv.$$

Since $\operatorname{Re} f(z) = u = \text{const.}$ it follows that $u_x = u_y = 0$. Therefore, since f is analytic, it follows from the Cauchy-Riemann equations that $v_x = v_y = 0$, and hence, $v = \text{const.}$ We conclude that $f(z) = \text{const.}$

5. Find the complex line integral

$$\oint_C \operatorname{Re} 4z \, dz,$$

where C is the closed curve consisting of the upper half of the unit circle from $z = 1$ to $z = -1$, and the straight segment connecting $z = -1$ to $z = 1$.

A. We write $C = C_1 + C_2$, where

$$C_1 : z(t) = e^{it}, \quad 0 \leq t \leq \pi, \quad \dot{z}(t) = ie^{it};$$

and

$$C_2 : z(t) = t, \quad -1 \leq t \leq 1, \quad \dot{z}(t) = 1.$$

On C_1 $\operatorname{Re} 4z = 4 \cos t$, hence

$$\begin{aligned} \int_{C_1} \operatorname{Re} z \, dz &= 4 \int_0^\pi \cos t (ie^{it}) \, dt = 4i \int_0^\pi (\cos^2 t + i \cos t \sin t) \, dt \\ &= 2 \int_0^\pi (i + i \cos 2t - \sin 2t) \, dt = 2\pi i. \end{aligned}$$

On C_2 $\operatorname{Re} z = 4t$, hence

$$\int_{C_2} \operatorname{Re} z \, dz = 4 \int_{-1}^1 t \, dt = 2t^2 \Big|_{-1}^1 = 0.$$

We conclude that

$$\oint_C \operatorname{Re} z \, dz = 2\pi i.$$

6. The ends of a laterally insulated bar are kept at zero temperature, the initial temperature is $f(x)$, $c = 1$, $L = 1$, and a heat source of constant energy output Q is present. The equation now is

$$u_t = u_{xx} + Q.$$

Find the temperature $u(x, t)$. (Try $u = v - Qx(x - 1)/2$. You may use, without deriving it, the sine series expansion of the solution for the problem with zero heat source).

A. We use the hint and note that $u_t = v_t$ and $u_{xx} = v_{xx} - Q$. Substituting these into the equation yields

$$v_t = v_{xx},$$

which is the heat equation for v without any sources. The boundary conditions are still zero, and the new initial condition for v is $f_N(x) = f(x) + Qx(x - 1)/2$. Then, the series solution is

$$v(x, t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) \exp(-n^2\pi^2 t).$$

The coefficients B_n are obtained from the Fourier series expansion of f_N , i.e.,

$$B_n = 2 \int_0^1 \left(f(x) + \frac{1}{2}Qx(x - 1) \right) \sin(n\pi x).$$

Therefore, the solution is

$$u(x, t) = v(x, t) - \frac{1}{2}Qx(x - 1) = -\frac{1}{2}Qx(x - 1) + \sum_{n=1}^{\infty} B_n \sin(n\pi x) \exp(-n^2\pi^2 t).$$

We note that the series decays as $\exp(-\pi^2 t)$ when $t \rightarrow \infty$, and the steady solution of the problem is

$$\bar{u}(x) = \frac{1}{2}Qx(1 - x).$$

7. Find the complex line integral

$$\oint_C \left(z + \frac{1}{z}\right) dz,$$

where C is the circle of radius 4 (counterclockwise).

A. We note that $f(z) = z$ is analytic so the integral is zero by the Cauchy Theorem.

Next, we parametrize $C : z(t) = 4e^{it}$ for $0 \leq t \leq 2\pi$. Then, $\dot{z}(t) = 4ie^{it}$ and

$$\oint_C \frac{1}{z} dz = \int_0^{2\pi} e^{-it} i e^{it} dt = i \int_0^{2\pi} dt = 2\pi i.$$

Thus,

$$\oint_C \left(z + \frac{1}{z}\right) dz = 2\pi i.$$

8. The ends of a long laterally insulated bar are kept at constant temperatures $u(0, t) = U_0$ and $u(L, t) = U_L$. The initial temperature is

$$u(x, 0) = f(x) = 10 \sin(\pi x/L) + (x/L)U_L + (1 - (x/L))U_0,$$

and $c = 1$. Find the temperature $u(x, t)$. What is the temperature as $t \rightarrow \infty$? (You may use, without deriving it, the sine series expansion of the solution for the problem with zero boundary conditions).

A. We use the hint and write $u(x, t) = v(x, t) + (x/L)U_L + (1 - (x/L))U_0$. Then, $u_t = v_t$ and $u_{xx} = v_{xx}$. Moreover, $U_0 = u(0, t) = v(0, t) + U_0$, hence $v(0, t) = 0$ and similarly $v(L, t) = 0$. Thus, v is a solution of $v_t = v_{xx}$ with zero boundary conditions. Also, initially $v(x, 0) = 10 \sin(\pi x/L)$.

Therefore, we may use the series representation of the solution, which in this case is just

$$v(x, t) = 10 \sin(\pi x/L) \exp\left(-\frac{c^2 \pi^2}{L^2} t\right).$$

We conclude that the solution is

$$u(x, t) = 10 \sin(\pi x/L) \exp\left(-\frac{c^2 \pi^2}{L^2} t\right) + \frac{x}{L} U_L + \left(1 - \frac{x}{L}\right) U_0.$$

When $t \rightarrow \infty$ we obtain the steady solution

$$\bar{u}(x) = \frac{x}{L} U_L + \left(1 - \frac{x}{L}\right) U_0.$$

9. (You have to answer this question) Find the following three complex line integrals and state the connection among them.

$$I_k = \int_{C_k} \frac{1}{z^2} dz,$$

for $k = 1, 2, 3$, where:

- (i) C_1 is the upper half of the circle of radius 2 and center at zero;
- (ii) C_2 is the lower half of the circle of radius 2 and center at zero;
- (iii) C_3 is the circle of radius 2 and center at zero.

A. We begin by parametrization of the three curves.

$$C_1 : z(t) = 2e^{it}, \quad 0 \leq t \leq \pi, \quad \dot{z}(t) = 2ie^{it}.$$

$$C_2 : z(t) = 2e^{it}, \quad \pi \leq t \leq 2\pi, \quad \dot{z}(t) = 2ie^{it}.$$

$$C_3 : z(t) = 2e^{it}, \quad 0 \leq t \leq 2\pi, \quad \dot{z}(t) = 2ie^{it}.$$

Then,

$$\begin{aligned} I_1 &= \int_{C_1} \frac{1}{z^2} dz = \int_0^\pi \frac{1}{4} e^{-2it} (2ie^{it}) dt = \frac{i}{2} \int_0^\pi e^{-it} dt \\ &= \frac{i}{2} \cdot \frac{1}{-i} e^{-it} \Big|_0^\pi = -\frac{1}{2} (e^{-i\pi} - 1) = 1. \end{aligned}$$

$$\begin{aligned} I_2 &= \int_{C_2} \frac{1}{z^2} dz = \int_\pi^{2\pi} \frac{1}{4} e^{-2it} (2ie^{it}) dt = \frac{i}{2} \int_\pi^{2\pi} e^{-it} dt \\ &= \frac{i}{2} \cdot \frac{1}{-i} e^{-it} \Big|_\pi^{2\pi} = -\frac{1}{2} (e^{-2\pi i} - e^{-i\pi}) = -1. \end{aligned}$$

Next,

$$\begin{aligned} I_3 &= \int_{C_3} \frac{1}{z^2} dz = \int_0^{2\pi} \frac{1}{4} e^{-2it} (2ie^{it}) dt = \frac{i}{2} \int_0^{2\pi} e^{-it} dt \\ &= \frac{i}{2} \cdot \frac{1}{-i} e^{-it} \Big|_0^{2\pi} = -\frac{1}{2} (1 - 1) = 0. \end{aligned}$$

We conclude, as expected since this is the *additive property of line integrals*, that

$$I_3 = I_1 + I_2.$$

10. (You have to answer this question) Show that among all rectangular membranes with fixed area $A = ab$ and the same c , the frequency of u_{22} (i.e., λ_{22}) is the smallest when the membrane is a square.

A. The expression for the eigenvalues of a rectangular membrane with sides a and b is

$$\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

Since we are interested in the minimum value we may drop the constants $c\pi$ and use the fact that $A = ab$ is given, to write

$$Y = \lambda_{22}^2 = \frac{4}{a^2} + \frac{4}{b^2} = \frac{4}{a^2} + \frac{4a^2}{A^2}.$$

At the minimum value $dY/da = 0$, thus,

$$0 = \frac{dY}{da} = -\frac{2}{a^3} + \frac{2a}{A^2}.$$

hence

$$a^4 = A^2 \implies a = \sqrt{A} \implies b = \sqrt{A}.$$

Moreover,

$$\frac{d^2Y}{da^2} = \frac{6}{a^4} + \frac{2}{A^2} > 0,$$

so it is a minimum value at $a = \sqrt{A}$.

We conclude that

$$a = b,$$

and so the frequency is the lowest when the membrane is a square.