

APM 542, Winter 2007
Exam 1, February 9

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You have 90 minutes and you have to answer 7 questions. Answer 5 out of questions 1–8, and you have to answer questions 9 and 10. Mark clearly which three questions are **not** to be graded. Each of the question 1–8 is worth 18 points and each of 9 and 10 is worth 15 points (total of 120). Show full logic for full credit. You may use one page written freely on one side.

Good luck!

1. Find the general solution of the system

$$y_1' = y_2, \quad y_2' = -y_1.$$

Find all the critical points and their stability. What is the shape of the trajectories?

A. The matrix for the system is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$, and the corresponding vectors are $\mathbf{v}_1 = (1, i)^T$ and $\mathbf{v}_2 = (1, -i)^T$. Therefore, the general solution is

$$\mathbf{y}(t) = (y_1(t), y_2(t))^T = c_1 \mathbf{v}_1 e^{it} + c_2 \mathbf{v}_2 e^{-it}.$$

Now, $p = 0$, $q = 1$ and $\Delta = -4$, and the origin $(0, 0)$, which is the only critical point, is a center. So it is stable, but not attracting.

The trajectories are circles. Indeed, multiply the first equation by y_1 , the second equation by y_2 , add the two expressions and integrate. Thus,

$$y_1' y_1 + y_2' y_2 = 0 \quad \implies \quad \frac{1}{2} y_1^2 + \frac{1}{2} y_2^2 = R^2.$$

2. Find the general solution of the following system and the type of the critical point,

$$\begin{aligned}y_1' &= 2y_1 + 2y_2, \\y_2' &= 5y_1 - y_2.\end{aligned}$$

What is the solution when $y_1(0) = 0$ and $y_2(0) = 1$?

A. The matrix for the system is

$$A = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -3$, and the corresponding vectors are $\mathbf{v}_1 = (1, 1)^T$ and $\mathbf{v}_2 = (1, -5/2)^T$. Therefore, the general solution is

$$\mathbf{y}(t) = (y_1(t), y_2(t))^T = c_1\mathbf{v}_1e^{4t} + c_2\mathbf{v}_2e^{-3t}.$$

Now, $p = 1$, $q = -12$ and $\Delta = 48$ and thus the origin $(0, 0)$, which is the only critical point, is a saddle point.

Finally, for the given initial conditions we have $c_1 + c_2 = 0$ and $c_1 - (5/2)c_2 = 1$, therefore, $c_1 = 2/7$ and $c_2 = -2/7$, so

$$\mathbf{y}(t) = (y_1(t), y_2(t))^T = \frac{2}{7}\mathbf{v}_1e^{4t} - \frac{2}{7}\mathbf{v}_2e^{-3t}.$$

3. Find the solution $u = u(x, t)$ of a string with $L = \pi$ and $c = 6$, when the initial velocity is zero and the initial deflection is

$$u(x, 0) = D \sin 9x.$$

A. The general solution, using separation of variables, is

$$u(x, t) = \sum_{n=1}^{n=\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin(nx),$$

where $\lambda_n = cn\pi/L = 6n$. By computing the B s we find that all the $B_n^* = 0$ since the initial velocity is zero, and all the $B_n = 0$ except for $B_9 = D$. Therefore, the solution is

$$u(x, t) = D \cos 54t \sin 9x.$$

Indeed, at $t = 0$ we have $u(x, 0) = D \sin 9x$.

4. Describe the trajectories in the phase plane of the Duffing equation

$$y'' + \omega_0^2 y - \beta y^3 = 0.$$

Assume that $|\beta|$ is small. When $\beta > 0$ it represents a *soft spring*, and when $0 > \beta$ it represents a *hard spring*. Can you explain the terms?

A. Let $y_1 = y$ and $y_2 = y'$, Then, the equation can be written as

$$\mathbf{y}' = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}' = \begin{pmatrix} y_2(t) \\ -\omega_0^2 y_1(t) + \beta y_1^3(t) \end{pmatrix}.$$

The critical points are $y_2 = 0$ and the roots of

$$y_1 (\omega_0^2 - \beta y_1^2) = 0.$$

This means that $y_1 = 0, \pm\sqrt{\omega_0^2/\beta}$. Thus, when $\beta > 0$ there are three critical points $(0, 0)$ and $(\pm\sqrt{\omega_0^2/\beta}, 0)$, while when $0 > \beta$ the origin is the only critical point. We let $\mu = \sqrt{\omega_0^2/\beta} = \omega_0/\sqrt{\beta}$, when $\beta > 0$.

Linearization of the system at $(0, 0)$ yields

$$\mathbf{y}' = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We read off the matrix A that $p = 0$ and $q = \det(A) = \omega_0^2$, and the point is a center.

We now investigate $(\mu, 0)$, the point $(-\mu, 0)$ has the same stability properties. We introduce the new variables $\bar{y}_1 = y_1 - \mu$ and $\bar{y}_2 = y_2$. We note that to the first order

$$-\omega_0^2(\bar{y}_1 + \mu) - \beta(\bar{y}_1 + \mu)^3 \approx -(\omega_0^2\mu + \beta\mu^3) - (\omega_0^2 + 3\mu^2\beta)\bar{y}_1 = 2\omega_0^2\bar{y}_1.$$

Since $\mu^2 = \omega_0^2/(-\beta)$.

Thus,

$$\bar{\mathbf{y}}' = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 2\omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}.$$

We read off the matrix A that $p = 0$ and $q = \det(A) = -2\omega_0^2$, and the point is a saddle point.

Therefore, the phase portrait consists of a center at the origin, and two saddle point points at $(\pm\mu, 0)$.

5. An long string (of length 202) is fixed at both ends,

$$u_{tt} - 25u_{xx} = 0, \quad -101 < x < 101,$$

has the initial displacement $u(x, 0) = f(x)$ and velocity $u_t(x, 0) = g(x) = 0$, where

$$f(x) = \begin{cases} 1 + x, & -1 \leq x \leq 0, \\ 1 - x, & 0 \leq x \leq 1, \\ 0, & 1 \leq |x|. \end{cases}$$

Find (i) $u(20, 3)$; (ii) $u(20, 4)$; (iii) $u(20, 10)$; (iv) $u(20, 25)$.

A. The speed of the waves in this problem is $c = 5$, and so on the time interval $0 \leq t \leq 20$ there are two waves starting from the initial condition and the boundary $u(-101, t) = u(101, t) = 0$ are satisfied. By the uniqueness of the solution we may use the D'Alembert solution, with $g(x) = 0$ and $c = 5$, thus

$$u(x, t) = \frac{1}{2} (f(x + 5t) + f(x - 5t)).$$

Therefore,

- (i) $u(20, 3) = \frac{1}{2} (f(20 + 15) + f(20 - 15)) = 0$
- (ii) $u(20, 4) = \frac{1}{2} (f(20 + 20) + f(20 - 20)) = 1$;
- (iii) $u(20, 10) = \frac{1}{2} (f(20 + 50) + f(20 - 50)) = 0$;
- (iv) $u(20, 25)$ - Here $20 < t$ and we cannot use the D'Alembert solution!.

6. Consider the wave equation

$$u_{tt} - 4u_{xx} = 0, \quad -\infty < x < +\infty.$$

Initially, the velocity is zero and the displacement is

$$f(x) = \begin{cases} 1 + x, & -1 \leq x \leq 0, \\ 1 - x, & 0 \leq x \leq 1, \\ 0, & 1 \leq |x|. \end{cases}$$

Find (i) $u(16, 7.6)$; (ii) $u(16, 7.9)$; (iii) $u(16, 8)$; (iv) $u(16, 9.2)$.

A. The D'Alembert solution, with $g(x) = 0$ and $c = 2$, is

$$u(x, t) = \frac{1}{2} (f(x + 2t) + f(x - 2t)).$$

Therefore,

$$\begin{aligned} (i) \quad u(16, 7.6) &= \frac{1}{2} (f(16 + 2 \times 7.6) + f(16 - 2 \times 7.6)) \\ &= \frac{1}{2} (f(31.2) + f(0.8)) = \frac{1}{2} (0 + 0.2) = 0.1. \end{aligned}$$

$$(ii) \quad u(16, 7.9) = \frac{1}{2} (0 + 0.8) = 0.4. \quad (iii) \quad u(16, 8) = \frac{1}{2} (0 + 1) = 0.5.$$

$$(iv) \quad u(16, 9.2) = \frac{1}{2} (0 + 0) = 0.$$

7. Describe the phase plane trajectories of the system

$$yy'' + 2(y')^2 = 0.$$

A. We divide by yy' and rewrite the equation as

$$\frac{y''}{y'} + 2\frac{y'}{y} = 0.$$

Recalling that $(\ln y)' = y'/y$ we obtain, upon integration

$$\ln y' + 2 \ln y = c.$$

Letting $y_1 = y$ and $y_2 = y'$ we obtain

$$\ln(y_2) + \ln(y_1^2) = \ln(y_2 y_1^2) = c,$$

where c is a constant. Therefore, $y_2 y_1^2 = c$, and so the trajectories in the phase plane are given by

$$y_2 = \frac{c}{y_1^2}.$$

8. Describe the phase plane portrait of the equation

$$y'' + y' - 4y + y^3 = 0.$$

A. Let $y_1 = y$ and $y_2 = y'$, Then the equation can be written as

$$\mathbf{y}' = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}' = \begin{pmatrix} y_2(t) \\ 4y_1(t) - y_1^3(t) - y_2(t) \end{pmatrix}.$$

The critical points are $y_2 = 0$ and

$$y_1(4 - y_1^2) = 0.$$

This means that $y_1 = 0, \pm 2$. Thus, there are three critical points $(0, 0)$ and $(\pm 2, 0)$.

Linearization of the system at $(0, 0)$ yields

$$\mathbf{y}' = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We read off the matrix A that $p = -1$ and $q = \det(A) = -4$, and the point is a saddle point.

We now investigate $(2, 0)$, the point $(-2, 0)$ has the same stability properties. We introduce the new variables $\bar{y}_1 = y_1 - 2$ and $\bar{y}_2 = y_2$. In terms of these variables the system is

$$\bar{\mathbf{y}}' = \begin{pmatrix} \bar{y}_1(t) \\ \bar{y}_2(t) \end{pmatrix}' = \begin{pmatrix} \bar{y}_2(t) \\ -\bar{y}_1^3(t) - 6\bar{y}_1^2(t) - 8\bar{y}_1(t) - \bar{y}_2(t) \end{pmatrix}.$$

After linearization, we drop powers of \bar{y}_1 , thus,

$$\bar{\mathbf{y}}' = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}' = A \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -8 & -1 \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}.$$

We read off the matrix A that $p = -1$ and $q = \det(A) = 8$, and the point is a spiral in.

Therefore, the phase portrait consists of a saddle point at the origin, and two stable spiral points at $(\pm 2, 0)$, with trajectories spiraling down to one or the other of the of the stable points.

9. (You have to answer this question) Consider the problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, & 0 < x < L, \\ u(0, t) &= 0, \\ -u_x(L, t) &= hu(L, t) \\ u(x, 0) &= f(x), & 0 \leq x \leq L \\ u_t(x, 0) &= g(x), & 0 \leq x \leq L. \end{aligned}$$

The energy of the system at time t is given by

$$E(t) = \frac{1}{2} \int_0^L u_t^2(x, t) dx + \frac{c^2}{2} \int_0^L u_x^2(x, t) dx + \frac{1}{2} c^2 h u^2(L, t).$$

Does the system conserves the energy?

A. We need to show that, if $u = u(x, t)$ is the solution, either

$$\frac{dE}{dt} = 0, \tag{1}$$

or equivalently, that

$$E(t) = E(0). \tag{2}$$

We show (1) first, and then (2), as each method provides a different insight.

So we compute E' first. We are allowed to interchange the integration and differentiation, thus

$$\frac{dE}{dt} = \frac{1}{2} \int_0^L \frac{\partial u_t^2(x, t)}{\partial t} dx + \frac{c^2}{2} \int_0^L \frac{\partial u_x^2(x, t)}{\partial t} dx + \frac{1}{2} c^2 h \frac{\partial u^2(L, t)}{\partial t}. \tag{3}$$

Now,

$$\frac{1}{2} \frac{\partial(u_t^2(x, t))}{\partial t} = u_t u_{tt}, \quad \frac{1}{2} \frac{\partial(u_x^2(x, t))}{\partial t} = u_x u_{tx}, \quad \frac{1}{2} \frac{\partial(u^2(L, t))}{\partial t} = u(L, t) u_t(L, t).$$

We insert these into (3) and obtain,

$$\frac{dE}{dt} = \int_0^L u_t u_{tt} dx + c^2 \int_0^L u_x u_{tx} dx + c^2 h u(L, t) u_t(L, t).$$

Next, we perform integration by parts in the second term on the right-hand side

$$c^2 \int_0^L u_x u_{tx} dx = -c^2 \int_0^L u_{xx} u_t dx + c^2 u_x u_t \Big|_0^L.$$

Now, from the problem we have that $u(0, t) = 0$ and so $u_t(0, t) = 0$, and $-u_x(L, t) = hu(L, t)$, thus, upon rearranging

$$\frac{dE}{dt} = \int_0^L u_t(u_{tt} - c^2 u_{xx}) dx - c^2 hu(L, t)u_t(L, t) + c^2 hu(L, t)u_t(L, t) = 0.$$

Here, we also used the fact that u satisfies the equation $u_{tt} - c^2 u_{xx} = 0$.

We conclude that the energy does not change, i.e., *it is conserved*.

In the second approach we show (2), and proceed in the reverse direction. We multiply the equation by u_t and integrate over $0 \leq t \leq T$ and over $0 \leq x \leq L$, and obtain

$$\int_0^T \int_0^L u_{tt} u_t dx dt - c^2 \int_0^T \int_0^L u_{xx} u_t dx dt = 0.$$

We observe that $(1/2)\partial(u_t^2)/\partial t = u_{tt}u_t$. Thus, integration by parts with respect to time in the first term, and using the initial condition $u_t(x, 0) = g(x)$ yields

$$\frac{1}{2} \int_0^L u_t^2(x, T) dx - \frac{1}{2} \int_0^L g^2(x) dx.$$

Next,

$$u_{xx}u_t = \frac{\partial}{\partial x}(u_x u_t) - u_x u_{tx} = \frac{\partial}{\partial x}(u_x u_t) - \frac{1}{2} \frac{\partial}{\partial t}(u_x^2).$$

Therefore, when we integrate over $0 \leq x \leq L$ the first term on the right-hand side we find $-c^2 u_x(L, t)u_t(L, t)$, and by using the boundary condition at $x = L$ we obtain $c^2 hu(L, t)u_t(L, t) = (c^2 h/2)\partial(u(L, t)^2)/\partial t$, and integration over $0 \leq t \leq T$ gives

$$\frac{c^2 h}{2} \int_0^T \frac{\partial(u(L, t)^2)}{\partial t} dt = \frac{c^2 h}{2} u^2(L, T) - \frac{c^2 h}{2} f^2(L),$$

since $u(L, 0) = f(L)$.

The integration of the second term, over $0 \leq t \leq T$, yields

$$-c^2 \int_0^T \int_0^L u_{xx} u_t \, dx dt = c^2 \int_0^L u_x^2(x, T) \, dx - c^2 \int_0^L (f')^2(x) \, dx,$$

since $u_x(x, 0) = f'(x)$. Combining all the terms above and rearranging we obtain

$$\begin{aligned} E(t) &= \frac{1}{2} \int_0^L u_t^2(x, T) \, dx + c^2 \int_0^L u_x^2(x, T) \, dx + \frac{c^2 h}{2} u^2(L, T) \\ &= \frac{1}{2} \int_0^L g^2(x) \, dx + c^2 \int_0^L (f'(x))^2 \, dx + \frac{c^2 h}{2} f^2(L) = E(0). \end{aligned}$$

We conclude that $E(t) = E(0)$, which means that the energy is conserved.

The first term in E is the kinetic energy, the second one is the stored elastic energy, and the third one is the stored energy at the boundary point.

10. (You have to answer this question) Describe the phase portrait, when (i) $c = 0$; (ii) $c \neq 0$, of the nonlinear oscillator:

$$y'' - cy' + ky \left(1 - \frac{L_0}{\sqrt{y^2 + L^2}} \right) = 0.$$

A. Let $y_1 = y$ and $y_2 = y'$, Then the equation can be written as

$$\mathbf{y}' = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}' = \begin{pmatrix} y_2(t) \\ -ky_1(t) \left(1 - \frac{L_0}{\sqrt{y_1^2 + L^2}} \right) + cy_2(t) \end{pmatrix}.$$

The critical points are $y_2 = 0$ and

$$ky_1 \left(1 - \frac{L_0}{\sqrt{y_1^2 + L^2}} \right) = 0.$$

Thus, there are three critical points $(0, 0)$ and $(\pm l, 0)$, where $l^2 = L_0^2 - L^2$.

(i) $c = 0$. We start with this case. First we check the origin $(0, 0)$. Linearizing about it, by neglecting powers greater than one, and setting $l^* = (L_0 - L)/L (> 0)$, yields

$$\mathbf{y}' = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ kl^* & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}.$$

The determinant is $\det(A - \lambda I) = 0$, which has the solutions $\lambda_1 = \sqrt{kl^*}$, $\lambda_2 = -\sqrt{kl^*}$. Then, $p = 0$ and $q = -kl^* < 0$ and it is a *saddle point*.

We consider now the critical point $(l, 0)$. We change variables to $\bar{y}_1 = y_1 - l$, $\bar{y}_2 = y_2$. Then, the second equation becomes

$$\begin{aligned} \bar{y}_2' &= -k(l + \bar{y}_1) \left(1 - \frac{L_0}{\sqrt{L^2 + (l + \bar{y}_1)^2}} \right) \\ &= -k(l + \bar{y}_1) \left(1 - \frac{L_0}{\sqrt{L^2 + l^2 + 2l\bar{y}_1 + \bar{y}_1^2}} \right). \end{aligned}$$

Now,

$$\begin{aligned}
\left(1 - \frac{L_0}{\sqrt{L^2 + l^2 + 2l\bar{y}_1 + \bar{y}_1^2}}\right) &\approx \left(1 - \frac{L_0}{\sqrt{L_0^2 + 2l\bar{y}_1}}\right) \\
&\approx \left(1 - \frac{1}{\sqrt{1 + 2l\bar{y}_1/L_0^2}}\right) \\
&\approx (1 - 1 + l\bar{y}_1/L_0^2) \\
&= \frac{l}{L_0^2}\bar{y}_1.
\end{aligned}$$

Here we used the facts that $l^2 = L_0^2 - L^2$ and $1/\sqrt{1+x} = 1 - x/2 + \dots$.

Thus,

$$\bar{y}_2' \approx -k(l + \bar{y}_1) \frac{l}{L_0^2} \bar{y}_1 \approx -\frac{kl^2}{L_0^2} \bar{y}_1.$$

So,

$$\begin{pmatrix} \bar{y}_1(t) \\ \bar{y}_2(t) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{kl^2}{L_0^2} & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}.$$

Now, $p = 0$ and $q = kl^2/L_0^2 > 0$ and the critical point is a *center*.

(ii) When $c < 0$ the center becomes a spiral in since the system has damping, and when $c > 0$ the center becomes a spiral out.