# APM 542, Winter 2005
# Exam 1, February 23

M. Shillor

You have 90 minutes and you have to answer 8 questions. Answer 6 out of questions 1–8, and you have to answer questions 9 and 10. Mark clearly which two questions are **not** to be graded. Each question is worth 12.5 points (total of 100). Show full logic for full credit. You may use one page written freely on one side. **Good luck!**

1. Find the general solution of the system

$$y_1' = y_2, \qquad y_2' = y_1.$$

Find all the critical points and their stability.

What is the solution when $y_1(0) = 0$ and $y_2(0) = 1$?

**A**. The matrix for the system is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$, and the corresponding vectors are $\mathbf{v}_1 = (1, 1)^T$ and $\mathbf{v}_2 = (1, -1)^T$. Therefore, the general solution is

$$\mathbf{y}(t) = (y_1(t), y_2(t))^T = c_1\mathbf{v}_1e^t + c_2\mathbf{v}_2e^{-t}.$$

Now, $p = 0$, $q = -1$ and $\Delta = 4$, and thus the origin $(0, 0)$, which is the only critical point, is a saddle point.

The solution, given the initial conditions is such that $c_1 = -c_2 = 1/2$, thus

$$\mathbf{y}(t) = (y_1(t), y_2(t))^T = \frac{1}{2}\mathbf{v}_1e^t - \frac{1}{2}\mathbf{v}_2e^{-t}.$$

2. Find the general solution of the following system and the type of the critical point,

$$y_1' = 2y_1 + 2y_2,$$
$$y_2' = 5y_1 - y_2.$$

**A**. The matrix for the system is

$$A = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -3$, and the corresponding vectors are $\mathbf{v}_1 = (1, 1)^T$ and $\mathbf{v}_2 = (1, -5/2)^T$. Therefore, the general solution is

$$\mathbf{y}(t) = (y_1(t), y_2(t))^T = c_1\mathbf{v}_1 e^{4t} + c_2\mathbf{v}_2 e^{-3t}.$$

Now, $p = 1$, $q = -12$ and $\Delta = 48$ and thus the origin $(0, 0)$, which is the only critical point, is a saddle point.

3. Find the solution $u = u(x, t)$ of a string attached at both ends, with $L = \pi$ and $c^2 = 25$, when the initial velocity is zero and the initial deflection is

$$u(x, 0) = D \sin 6x.$$

**A**. The general solution, using separation of variables, is

$$u(x, t) = \sum_{n=1}^{n=\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin(nx),$$

where $\lambda_n = cn\pi/L = 5n$. By computing the $B$s we find that all the $B_n^* = 0$ since the initial velocity is zero, and all the $B_n = 0$ except for $B_6 = D$. Therefore, the solution is

$$u(x, t) = D \cos 30t \sin 6x.$$

Indeed, at $t = 0$ we have $u(x, 0) = D \sin 6x$.

4. Write the model for the temperature distribution $u$ in a body which occupies a domain $D$ in three dimensions, and has its outer surface divided into three non-overlapping parts $S_D$, $S_N$, and $S_R$. Now, the temperature on $S_D$ is $u_D$, the heat flux $q_N$ is prescribed on $S_N$, and the body exchanges heat with the environment over $S_R$. The outside temperature is $u_E = 0$, and the heat exchange coefficient is $h$.

**A**. The model is as follows.

$$\begin{aligned} u_t - c^2 \Delta u &= 0, && \text{in } D,\ t > 0, \\ u &= u_0, && \text{in } D,\ t = 0, \\ u &= u_D, && \text{on } S_D,\ t > 0, \\ -k\frac{\partial u}{\partial n} &= q_N, && \text{on } S_N,\ t > 0, \\ -k\frac{\partial u}{\partial n} + hu &= 0, && \text{on } S_R,\ t > 0. \end{aligned}$$

5. Describe the phase portrait, when (i) $c = 0$; (ii) $c \neq 0$, of the nonlinear oscillator:

$$y'' + cy' + ky\left(1 - \frac{L_0}{\sqrt{y^2 + L^2}}\right) = 0.$$

**A**. Let $y_1 = y$ and $y_2 = y'$, Then the equation can be written as

$$\mathbf{y}' = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}' = \begin{pmatrix} y_2(t) \\ -ky_1(t)\left(1 - \frac{L_0}{\sqrt{y^2+L^2}}\right) - cy_2(t) \end{pmatrix}.$$

The critical points are $y_2 = 0$ and

$$ky_1\left(1 - \frac{L_0}{\sqrt{y_1^2 + L^2}}\right) = 0.$$

Thus, there are three critical points $(0, 0)$ and $(\pm l, 0)$, where $l^2 = L_0^2 - L^2$.

(i) $c = 0$. We start with this case. First we check the origin $(0,0)$. Linearizing about it, by neglecting powers greater than one, and setting $l^* = (L_0 - L)/L\ (> 0)$, yields

$$\mathbf{y}' = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ kl^* & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}.$$

The determinant is $\det(A - \lambda I) = 0$, which has the solutions $\lambda_1 = \sqrt{kl^*}, \lambda_2 = -\sqrt{kl^*}$. Then, $p = 0$ and $q = -kl^* < 0$ and it is a *saddle point*.

We consider now the critical point $(l, 0)$. We change variables to $\bar{y}_1 = y_1 - l, \bar{y}_2 = y_2$. Then, the second equation becomes

$$\begin{aligned} \bar{y}_2' &= -k(l + \bar{y}_1)\left(1 - \frac{L_0}{\sqrt{L^2 + (l + \bar{y}_1)^2}}\right) \\ &= -k(l + \bar{y}_1)\left(1 - \frac{L_0}{\sqrt{L^2 + l^2 + 2l\bar{y}_1 + \bar{y}_1^2}}\right). \end{aligned}$$

Now,

$$\begin{aligned} \left(1 - \frac{L_0}{\sqrt{L^2 + l^2 + 2l\bar{y}_1 + \bar{y}_1^2}}\right) &\approx \left(1 - \frac{L_0}{\sqrt{L_0^2 + 2l\bar{y}_1}}\right) \\ &\approx \left(1 - \frac{1}{\sqrt{1 + 2l\bar{y}_1/L_0^2}}\right) \\ &\approx \left(1 - 1 + l\bar{y}_1/L_0^2\right) \\ &= \frac{l}{L_0^2}\bar{y}_1. \end{aligned}$$

Here we used the facts that $l^2 = L_0^2 - L^2$ and $1/\sqrt{1+x} = 1 - x/2 + \ldots$.

Thus,

$$\bar{y}_2' \approx -k(l + \bar{y}_1)\frac{l}{L_0^2}\bar{y}_1 \approx -\frac{kl^2}{L_0^2}\bar{y}_1.$$

So,

$$\begin{pmatrix} \bar{y}_1(t) \\ \bar{y}_2(t) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{kl^2}{L_0^2} & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}.$$

Now, $p = 0$ and $q = \frac{kl^2}{L_0^2} > 0$ and the critical point is a *center*.

(ii) When $c > 0$ the center becomes a spiral in.

6. Consider the wave equation

$$u_{tt} - 4u_{xx} = 0, \qquad -\infty < x < +\infty.$$

Initially, the velocity is zero and the displacement is

$$f(x) = \begin{cases} 1+x, & -1 \le x \le 0, \\ 1-x, & 0 \le x \le 1, \\ 0, & 1 \le |x|. \end{cases}$$

Find (i) $u(16, 7.6)$; (ii) $u(16, 7.9)$; (iii) $u(16, 8)$; (iv) $u(16, 9.2)$.

**A**. The D'Alembert solution, with $g(x) = 0$ and $c = 2$, is

$$u(x,t) = \frac{1}{2}\left(f(x+2t) + f(x-2t)\right).$$

Therefore,

$$\begin{aligned} (i) \qquad u(16, 7.6) &= \frac{1}{2}\left(f(16 + 2\times 7.6) + f(16 - 2\times 7.6)\right) \\ &= \frac{1}{2}(f(31.2 + f(0.8) = \frac{1}{2}(0 + 0.2) = 0.1. \end{aligned}$$

$$(ii) \qquad u(16, 7.9) = \frac{1}{2}(0 + 0.8) = 0.4. \qquad (iii) \qquad u(16, 8) = \frac{1}{2}(0+1) = 0.5.$$

$$(iv) \qquad u(16, 9.2) = \frac{1}{2}(0+0) = 0.$$

7. The beam equation is

$$u_{tt} + 4u_{xxxx} = 0.$$

The beam is clamped at $x = 0$, so $u(0,t) = u_x(0,t) = 0$. At $x = L$ it satisfies $u(L,t) = u_{xx}(L,t) = 0$. Find the equation for the eigenvalues $(\beta L)$.

**A**. we use separation of variables, so let $u(x,t) = F(x)G(t)$. Then, it follows from the equation that

$$F\ddot{G} + 4F''''G = 0.$$

Therefore,

$$-\frac{\ddot{G}}{4G} = \frac{F''''}{F} = -\beta^4 = const.$$

The solutions are

$$F(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x,$$

$$G(t) = a\cos c\beta^2 t + b\sin c\beta^2 t$$

The boundary conditions at $x = 0$ yield $F(0) = F'(0) = 0$, and $F(L) = F''(L) = 0$. The first two lead to $C = -A$ and $D = -B$. The other two yield

$$F(x) = A\cos\beta x + B\sin\beta x - A\cosh\beta x - B\sinh\beta x,$$

$$F''(x) = -\beta^2 A\cos\beta x - \beta^2 B\sin\beta x - \beta^2 A\cosh\beta x - \beta^2 B\sinh\beta x.$$

Then, $F(L) = 0$ and $F''(L) = 0$ yield

$$0 = A\cos\beta L + B\sin\beta L - A\cosh\beta L - B\sinh\beta L,$$

$$0 = -\beta^2 A\cos\beta L - \beta^2 B\sin\beta L - \beta^2 A\cosh\beta L - \beta^2 B\sinh\beta L.$$

These lead to

$$0 = A\cos\beta L + B\sin\beta L, \qquad 0 = A\cosh\beta L + B\sinh\beta L.$$

Then, the required condition is

$$\tanh\beta L = \tan\beta L.$$

This equation has an infinite number of solutions, leading to the Fourier representation of the solution of the problem.

8. Describe the phase plane portrait of the equation

$$y'' + y' - 4y + y^3 = 0.$$

**A**. Let $y_1 = y$ and $y_2 = y'$, Then the equation can be written as

$$\mathbf{y}' = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}' = \begin{pmatrix} y_2(t) \\ 4y_1(t) - y_1^3(t) - y_2(t) \end{pmatrix}.$$

The critical points are $y_2 = 0$ and

$$y_1\left(4 - y_1^2\right) = 0.$$

This means that $y_1 = 0, \pm 2$. Thus, there are three critical points $(0, 0)$ and $(\pm 2, 0)$.

Linearization of the system at $(0, 0)$ yields

$$\mathbf{y}' = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We read off the matrix $A$ that $p = -1$ and $q = \det(A) = -4$, and the point is a saddle point.

We now investigate $(2, 0)$, the point $(-2, 0)$ has the same stability properties. We introduce the new variables $\bar{y}_1 = y_1 - 2$ and $\bar{y}_2 = y_2$. In terms of these variables the system is

$$\bar{\mathbf{y}}' = \begin{pmatrix} \bar{y}_1(t) \\ \bar{y}_2(t) \end{pmatrix}' = \begin{pmatrix} \bar{y}_2(t) \\ -\bar{y}_1^3(t) - 6\bar{y}_1^2(t) - 8\bar{y}_1(t) - \bar{y}_2(t) \end{pmatrix}.$$

After linearization, we drop powers of $\bar{y}_1$, thus,

$$\bar{\mathbf{y}}' = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}' = A\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -8 & -1 \end{pmatrix}\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}.$$

We read off the matrix $A$ that $p = -1$ and $q = \det(A) = 8$, and the point is a spiral in.

Therefore, the phase portrait consists of a saddle point at the origin, and two stable spiral points at $(\pm 2, 0)$, with trajectories spiraling down to one or the other of the of the stable points.

9. (You have to answer this question) Consider the problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, & 0 < x < L, \\ u(0,t) &= 0, & \\ -u_x(L,t) &= hu(L,t) & \\ u(x,0) &= f(x), & 0 \le x \le L \\ u_t(x,0) &= g(x), & 0 \le x \le L. \end{aligned}$$

The energy of the system at time $t$ is given by

$$E(t) = \frac{1}{2}\int_0^L u_t^2(x,t)\,dx + \frac{c^2}{2}\int_0^L u_x^2(x,t)\,dx + \frac{1}{2}c^2 h u^2(L,t).$$

Does the system conserves the energy?

**A**. We need to show that, if $u = u(x,t)$ is the solution, either

$$\frac{dE}{dt} = 0, \tag{1}$$

or equivalently, that

$$E(t) = E(0). \tag{2}$$

We show (1) first, and then (2), as each method provides a different insight.

So we compute $E'$ first. We are allowed to interchange the integration and differentiation, thus

$$\frac{dE}{dt} = \frac{1}{2}\int_0^L \frac{du_t^2(x,t)}{dt}\,dx + \frac{c^2}{2}\int_0^L \frac{du_x^2(x,t)}{dt}\,dx + \frac{1}{2}c^2 h \frac{du^2(L,t)}{dt}. \tag{3}$$

Now,

$$\frac{1}{2}\frac{d(u_t^2(x,t))}{dt} = u_t u_{tt}, \quad \frac{1}{2}\frac{d(u_x^2(x,t))}{dt} = u_x u_{tx}, \quad \frac{1}{2}\frac{d(u^2(L,t))}{dt} = u(L,t)u_t(L,t).$$

We insert these into (3) and obtain,

$$\frac{dE}{dt} = \int_0^L u_t u_{tt}\,dx + c^2\int_0^L u_x u_{tx}\,dx + c^2 h u(L,t)u_t(L,t).$$

Next, we perform integration by parts in the second term on the right-hand side

$$c^2 \int_0^L u_x u_{tx}\, dx = -c^2 \int_0^L u_{xx} u_t\, dx + c^2 u_x u_t|_0^L.$$

Now, from the problem we have that $u(0,t) = 0$ and so $u_t(0,t) = 0$, and $-u_x(L,t) = hu(L,t)$, thus, upon rearranging

$$\frac{dE}{dt} = \int_0^L u_t(u_{tt} - c^2 u_{xx})\, dx - c^2 hu(L,t)u_t(L,t) + c^2 hu(L,t)u_t(L,t) = 0.$$

Here, we also used the fact that $u$ satisfies the equation $u_{tt} - c^2 u_{xx} = 0$.

We conclude that the energy does not change, i.e., *it is conserved.*

In the second approach we show (2), and proceed in the reverse direction. We multiply the equation by $u_t$ and integrate over $0 \le t \le T$ and over $0 \le x \le L$, and obtain

$$\int_0^T \int_0^L u_{tt} u_t\, dxdt - c^2 \int_0^T \int_0^L u_{xx} u_t\, dxdt = 0.$$

We observe that $(1/2)\partial(u_t^2)/\partial t = u_{tt}u_t$. Thus, integration by parts with respect to time in the first term, and using the initial condition $u_t(x,0) = g(x)$ yields

$$\frac{1}{2}\int_0^L u_t^2(x,T)\, dx - \frac{1}{2}\int_0^L g^2(x)\, dx.$$

Next,

$$u_{xx}u_t = \frac{\partial}{\partial x}(u_x u_t) - u_x u_{tx} = \frac{\partial}{\partial x}(u_x u_t) - \frac{1}{2}\frac{\partial}{\partial t}\left(u_x^2\right).$$

Therefore, when we integrate over $0 \le x \le L$ the first term on the right-hand side we find $-c^2 u_x(L,t)u_t(L,t)$, and by using the boundary condition at $x = L$ we obtain $c^2 hu(L,t)u_t(L,t) = (c^2 h/2)\partial(u(L,t)^2)/\partial t$, and integration over $0 \le t \le T$ gives

$$\frac{c^2 h}{2}\int_0^T \frac{\partial(u(L,t)^2)}{\partial t}\, dt = \frac{c^2 h}{2} u^2(L,T) - \frac{c^2 h}{2} f^2(L),$$

since $u(L,0) = f(L)$.

The integration of the second term, over $0 \le t \le T$, yields

$$-c^2 \int_0^T \int_0^L u_{xx} u_t \, dx dt = c^2 \int_0^L u_x^2(x,T) \, dx - c^2 \int_0^L (f')^2(x) \, dx,$$

since $u_x(x,0) = f'(x)$. Combining all the terms above and rearranging we obtain

$$E(t) = \frac{1}{2} \int_0^L u_t^2(x,T) \, dx + c^2 \int_0^L u_x^2(x,T) \, dx + \frac{c^2 h}{2} u^2(L,T)$$

$$= \frac{1}{2} \int_0^L g^2(x) \, dx + c^2 \int_0^L (f'(x))^2 \, dx + \frac{c^2 h}{2} f^2(L) = E(0).$$

We conclude that $E(t) = E(0)$, which means that the energy is conserved.

The first term in $E$ is the kinetic energy, the second one is the stored elastic energy, and the third one is the stored energy at the boundary point.

10. (You have to answer this question) Describe the trajectories in the phase plane of the Duffing equation

$$y'' + \omega_0^2 y - \beta y^3 = 0.$$

Assume that $|\beta|$ is small. When $\beta > 0$ it represents a *soft spring*, and when $0 > \beta$ it represents a *hard spring*. Can you explain the terms?

**A**. Let $y_1 = y$ and $y_2 = y'$, Then the equation can be written as

$$\mathbf{y}' = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}' = \begin{pmatrix} y_2(t) \\ -\omega_0^2 y_1(t) + \beta y_1^3(t) \end{pmatrix}.$$

The critical points are $y_2 = 0$ and the roots of

$$y_1\left(\omega_0^2 - \beta y_1^2\right) = 0.$$

This means that $y_1 = 0, \pm\sqrt{\omega_0^2/\beta}$. Thus, when $\beta > 0$ there are three critical points $(0,0)$ and $(\pm\sqrt{\omega_0^2/\beta}, 0)$, while when $0 > \beta$ the origin is the only critical point. We let $\mu = \sqrt{\omega_0^2/\beta} = \omega_0/\sqrt{\beta}$, when $\beta > 0$.

Linearization of the system at $(0,0)$ yields

$$\mathbf{y}' = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We read off the matrix $A$ that $p = 0$ and $q = \det(A) = \omega_0^2$, and the point is a center.

We now investigate $(\mu, 0)$, the point $(-\mu, 0)$ has the same stability properties. We introduce the new variables $\bar{y}_1 = y_1 - \mu$ and $\bar{y}_2 = y_2$. We note that to the first order

$$-\omega_0^2(\bar{y}_1 + \mu) - \beta(\bar{y}_1 + \mu)^3 \approx -(\omega_0^2\mu + \beta\mu^3) - (\omega_0^2 + 3\mu^2\beta)\bar{y}_1 = 2\omega_0^2\bar{y}_1.$$

Since $\mu^2 = \omega_0^2/(-\beta)$.

Thus,

$$\bar{\mathbf{y}}' = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 2\omega_0^2 & 0 \end{pmatrix}\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}.$$

We read off the matrix $A$ that $p = 0$ and $q = \det(A) = -2\omega_0^2$, and the point is a saddle point.

Therefore, the phase portrait consists of a center at the origin, and two saddle point points at $(\pm\mu, 0)$.