APM 542, Winter 2004
Exam 1 -Answers - February 18

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You have 90 minutes and you have to answer 8 questions. Answer 6 out of questions 1–8, and you have to answer questions 9 and 10. Mark clearly which two questions are **not** to be graded. Each question is worth 12.5 points (total of 100). Show full logic for full credit. You may use one page written freely on both sides. **Good luck!**

1. Find the general solution of the system

$$y_1' = -2y_1 - y_2, \qquad y_2' = y_1 - 2y_2.$$

Find all the critical points and their stability.

What is the solution when $y_1(0) = 0$ and $y_2(0) = 1$?

A: We have

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

hence $p = -4$, $q = 5$ and $\Delta = -4$. Therefore, the origin is an asymptotically stable spiral in.

To find the solution one can find the eigenvectors, but an easier way is to note that $p = -4 = \lambda_1 + \lambda_2$ and $q = 5 = \lambda_1\lambda_2$, so that $\lambda_{1,2} = -2 \pm i$. Then we assume that

$$\begin{aligned} y_1 &= Ae^{-2t}\cos t + Be^{-2t}\sin t, \\ y_2 &= Ce^{-2t}\cos t - De^{-2t}\sin t, \end{aligned}$$

and it follows from the initial conditions that $A = 0$ and $C = 1$. Then substituting the expressions into the equations yields $B = -1$ and $D = 0$, thus, the solution is

$$\begin{aligned} y_1 &= -e^{-2t}\sin t, \\ y_2 &= e^{-2t}\cos t. \end{aligned}$$

2. Find the general solution of the following system and the type of the critical point,

$$y_1' = 2y_1 + 2y_2,$$
$$y_2' = 5y_1 - y_2.$$

A: We have now

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

hence $p = 1$, $q = -12$ and, therefore, the origin is a saddle point.

To find the solution the easiest way is to note that $p = 1 = \lambda_1 + \lambda_2$ and $q = -12 = \lambda_1\lambda_2$, so that $\lambda_1 = 4$ and $\lambda_2 = -3$. Then let

$$y_1 = Ae^{4t} + Be^{-3t}, \qquad y_2 = Ce^{4t} + De^{-3t},$$

and by substituting the expressions into the equations we obtain $C = A$ and $D = -5B/2$, thus, the general solution is

$$\begin{aligned} y_1 &= Ae^{4t} + Be^{-3t}, \\ y_2 &= Ae^{4t} - \frac{5}{2}Be^{-3t}. \end{aligned}$$

The coefficents A and B are to be determined from the initial conditions.

3. Find the solution $u = u(x,t)$ of a string with $L = 2\pi$ and $c^2 = 25$, when the initial velocity is zero and the initial deflection is

$$u(x,0) = A\sin 10x.$$

A: We assume that the string is fixed at both ends. Then, using separation of variables we obtain that $p_n = n\pi/L = n/2$, and the eigenvalues are $\lambda_n = cn\pi/L = 5n/2$. Thus, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos(5nt/2) + B_n^* \sin(5nt/2)) \sin(nx/2).$$

It follows now from the initial condition that $B_{20} = A$ is the only coefficient that is not zero, therefore, the solution of the problem is

$$u(x,t) = A\cos(50t)\sin(10x).$$

4. Consider the heat equation

$$u_t - 9u_{xx} = 0, \qquad -\infty < x < +\infty.$$

Initially, the temperature is

$$f(x) = \begin{cases} 1+x, & -1 \le x \le 0, \\ 1-x, & 0 \le x \le 1, \\ 0, & 1 \le |x|. \end{cases}$$

Obtain an expression for $u(x,t)$. Write down $u(12,6)$ (do not compute it!). When is the solution nonzero at $x = 120$?
A: We have that $c = 3$ and for this f, using (11) in Section 11.6, that

$$u(x,t) = \frac{1}{6\sqrt{\pi t}} \left(\int_{-1}^{0} (1+s) \exp\left(-\frac{(x-s)^2}{36t} \right) ds \right.$$
$$\left. + \int_{0}^{1} (1-s) \exp\left(-\frac{(x-s)^2}{36t} \right) ds \right).$$

Now, $u(12,6)$ is obtained from this expression when $x = 12$ and $t = 6$. Since the two integrands are nonnegative and do not vanish identically, we have that $u(120,t) > 0$ for each $t > 0$, no matter how small t is. Indeed, the speed of propagation is infinite!

5. Consider the dynamic membrane problem over the rectangle $0 \le x \le a$, $0 \le y \le b$, with zero boundary conditions and initial conditions $u = f(x,y)$ and $u_t = g(x,y)$. Thus, we seek a function $u = u(x,y,t)$ such that

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\
u &= 0, \qquad \text{on the boundary} \\
u(x,y,0) &= f(x,y), \\
u_t(x,y,0) &= g(x,y).
\end{aligned}$$

i) How do the frequencies of the solution depend on the tension? On the mass density?

ii) How would you change the model if there is an obstacle $\phi(x,y)$ underneath it?

A: (i) We proceed with the separation of variables and obtain from the boundary conditions that

$$F_{mn} = \sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b}),$$

for $m, n = 1, 2, 3, \ldots$, and the vibration frequencies are

$$\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

Since the λ_{mn} are proportional to c and $c^2 = T/\rho$ we obtain that the frequencies depend on $\sqrt{T}$ and on $1/\sqrt{\rho}$.

(ii) To accommodate an obstacle underneath we change the equation to

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \xi,$$

where $\xi = \xi(x,y,t)$ is the reaction force that the obstacle exerts on the membrane to prevent it from penetrating it. Thus, we must require

$$\phi(x,y) \le u(x,y,t), \qquad 0 \le \xi(x,y,t), \qquad (u(x,y,t) - \phi(x,y))\xi(x,y,t) = 0.$$

The boundary conditions and the initial conditions are the same, except for the requirement

$$\phi(x,y) \le f(x,y)$$

which means that initially the membrane is not penetrating the obstacle.

6. Consider the wave equation

$$u_{tt} - 9u_{xx} = 0, \qquad -\infty < x < +\infty.$$

Initially, the velocity is zero and the displacement is

$$f(x) = \begin{cases} 1+x, & -1 \le x \le 0, \\ 1-x, & 0 \le x \le 1, \\ 0, & 1 \le |x|. \end{cases}$$

Find (i) $u(15, 3.6)$; (ii) $u(15, 4.2)$; (iii) $u(15, 5)$; (iv) $u(15, 9.2)$.

A: D'Alembert's solution for the problem is

$$u(x,t) = \frac{1}{2}(f(x+3t) + f(x-3t)),$$

since $c = 3$. To obtain the solution at the indicated points we just substitute the x and t valuse, thus

(i) $u(15, 3.6) = (f(20.8) + f(4.2))/2 = 0$;

(ii) $u(15, 4.2) = (f(27.6) + f(2.4))/2 = 0$;

(iii) $u(15, 5) = (f(30) + f(0))/2 = \frac{1}{2}$;

(iv) $u(15, 9.2) = (f(42.6) + f(-12.6))/2 = 0$.

7. The beam equation is $u_{tt} + 9u_{xxxx} = 0$. The beam is clamped at $x = 0$ and free at $x = L$. Derive the equation for the eigenvalues (wavelengths). What are the corresponding frequencies?

A: We use separation of variables $u(x,t) = F(x)G(t)$, then we have $F(0) = F'(0) = F''(L) = F'''(L) = 0$, and

$$\begin{aligned} \frac{F''''}{F} &= -\frac{\ddot{G}}{9G} = \beta^4, \\ F(x) &= A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x, \\ G(t) &= a\cos 3\beta^2 t + b\sin 3\beta^2 t. \end{aligned}$$

It follows from the boundary conditions $F(0) = F'(0) = 0$ that

$$C = -A, \qquad D = -B.$$

Next, the boundary conditions at $x = L$ mean that $F''(L) = F'''(L) = 0$, thus

$$\begin{aligned} F''(x) &= -\beta^2 \left(A\cos\beta x + B\sin\beta x + A\cosh\beta x + B\sinh\beta x\right), \\ F'''(x) &= \beta^3 \left(A\sin\beta x - B\sin\beta x - A\cosh\beta x - B\sinh\beta x\right). \end{aligned}$$

Then $F''(L) = F'''(L) = 0$ imply that

$$\begin{aligned} A(\cos\beta L + \cosh\beta L) &+ B(\sin\beta L + \sinh\beta L) = 0, \\ A(\sin\beta L - \sinh\beta L) &- B(\cos\beta L + \cosh\beta L) = 0. \end{aligned}$$

Dividing the two expressions and performing simple manipulations, using the identities $\cos^2\theta + \sin^2\theta = 1$ and $\cosh^2\theta - \sinh^2\theta = 1$, lead to

$$\cosh\beta L \cos\beta L = -1.$$

Writing this expression as

$$\cos\beta L = \frac{-1}{\cosh\beta L},$$

we can see by graphing both sides that there is an infinite sequence of intersection points which are the solutions β_n for $n = 1, 2, 3, \ldots$.

The frequencies are $\lambda_n = 3\beta_n^2$.

8. The longitudinal vibrations of a rod fixed at $x = 0$ and free at $x = L$ are described by

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, \qquad 0 < x < L, \\ u(0,t) &= 0, \\ u_x(L,t) &= 0, \\ u(x,0) &= f(x), \qquad 0 \le x \le L, \\ u_t(x,0) &= 0, \qquad 0 \le x \le L. \end{aligned}$$

Here $c^2 = E/\rho$, where E is the Young modulus and ρ the material density. Find the equation for the eigenfrequencies. How do they depend on E?

A: Using separation of variables $u(x,t) = F(x)G(t)$ and the boundary conditions yield $F(0) = 0$ and $F'(L) = 0$. Thus, if we let

$$F(x) = A\cos px + B\sin px,$$

then

$$F(0) = 0 \implies A = 0, \quad \text{and} \quad F'(L) = 0 \implies \cos pL = 0.$$

Therefore

$$p = p_n = \frac{(n + \frac{1}{2})\pi}{L}, \qquad n = 0, 1, 2, \ldots,$$

and

$$F_n(x) = B_n \sin \frac{(n + \frac{1}{2})\pi}{L} x,$$

for $n = 0, 1, 2, \ldots$.

The frequencies are given by

$$\lambda_n = cp_n = \frac{(n + \frac{1}{2})c\pi}{L}, \qquad n = 0, 1, 2, \ldots$$

Since λ_n is proportional to c then λ_n is proportional to $\sqrt{E}$.

9. (You have to answer this question) Consider the problem

$$\begin{aligned} u_{tt}-c^2u_{xx} &= 0, \qquad 0<x<L,\\ u(0,t) &= 0,\\ u_x(L,t) &= 0,\\ u(x,0) &= f(x), \qquad 0\le x\le L,\\ u_t(x,0) &= 0, \qquad 0\le x\le L. \end{aligned}$$

Can it have two different solutions? Explain carefully.
A: Assume that u_1 and u_2 are two solutions. Let $w(x,t)=u_1(x,t)-u_2(x,t)$. Then w satisfies the same problem as each one of the us, but with zero initial condition. We now multiply the equation for w by w_t and obtain

$$w_{tt}w_t - c^2 w_{xx}w_t = 0.$$

We integrate this expression over $0\le t\le T$ and over $0\le x\le L$, and since $w_{tt}w_t=(1/2)(w_t^2)_t$, we find

$$\frac{1}{2}\int_0^T\int_0^L (w_t^2)_t dxdt - c^2\int_0^T\int_0^L w_{xx}w_t dxdt = 0.$$

The integration in time of the first term on the left-hand side gives

$$\frac{1}{2}\int_0^L w_t^2(x,T)dx - \frac{1}{2}\int_0^L w_t^2(x,0)dx = \frac{1}{2}\int_0^L w_t^2(x,T)dx$$

since initially $w_t(x,0)=0$.

Next,

$$w_{xx}w_t = (w_xw_t)_x - w_xw_{xt} = (w_xw_t)_x - \frac{1}{2}(w_x^2)_t.$$

Integration of the term $(w_xw_t)_x$ over $0\le x\le L$ and using the boundary conditions $w_t(0,t)=w_t(L,t)=0$ shows that the term vanishes. Then,

$$-\frac{1}{2}\int_0^T\int_0^L (w_x^2)_t dxdt = -\frac{1}{2}\int_0^L w_x^2(x,T)dx,$$

since initially $w_x(x,0)=0$. Collecting the two expression yields

$$E_k(T)+E_p(T) = \frac{1}{2}\int_0^L w_t^2(x,T)dx + \frac{c^2}{2}\int_0^L w_x^2(x,T)dx = 0.$$

We conclude that $w = cost.$ and then by the boundary conditions $w=0$, hence $u_1=u_2$ and the solution is unique.

10. (You have to answer this question) Find and describe the phase plane trajectories of the equation

$$yy'' + (y')^2 = 1.$$

What is the solution if $y(0) = y_0$ and $y'(0) = v_0$?

A: We note that

$$yy'' + (y')^2 = \frac{d}{dt}(yy') = 1.$$

Then, integration in time yields

$$yy' = t + c.$$

Writing $y_1 = y$ and $y_2 = y'$ we obtain the solution curves in the phase plane

$$y_2 = \frac{t+c}{y_1}.$$

Next, we note that $yy' = (y^2)'/2$, and so another integration yields

$$y^2 = t^2 + ct + d^2,$$

where d^2 in another integration constant.

We conclude that the solutions are given by

$$y = \pm\sqrt{t^2 + ct + d^2}.$$

When $t = 0$ we have

$$y(0) = y_0 = \pm\sqrt{d^2} \implies d = \pm y_0.$$

Next, we have that

$$y' = \frac{t+c}{y} = \frac{t+c}{\pm\sqrt{t^2 + ct + y_0^2}},$$

and initially $y'(0) = v_0$, hence $v_0^2 = c^2/y_0^2$, and thus $c = \pm v_0 y_0$. We conclude that there are four solutions, and they are given by

$$y(t) = \pm\sqrt{t^2 \pm v_0 y_0 t + y_0^2}.$$

We note that the solution is NOT unique, there are four of them, and that the solutions may vanish at finite time! Moreover, there are no critical points.