APM-541 Fall 2004
EXAM 2-A- Solutions

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You have 90 minutes. Answer 5 questions out of 1–8, and answer questions 9 and 10. Mark clearly which questions are **not** to be graded. Each question 1–8 is worth 20 points and questions 9 and 10 are worth 10 points each (total of 120). You may use a two sided page freely written, and please attach the page to the exam. Show full logic for full credit.

Good luck!

1. Write a parametric representation of the curve C and find the tangent vector to the curve at the point P, where

$$(x-a)^2 + 4y^2 = 1, \qquad z = x^2; \qquad P(a+1, 0, (a+1)^2).$$

Here, a is a constant.

A: We choose $x = a + \cos t$, $y = \frac{1}{2}\sin t$, and $z = (a+\cos t)^2$. Then, a parametric representation of the curve is

$$\mathbf{r}(t) = (a+\cos t)\mathbf{i} + \frac{1}{2}\sin t\,\mathbf{j} + (a+\cos t)^2\mathbf{k}.$$

At the point P $x = a+1$, $y = 0$, and $z = (a+1)^2$, so that the value of the parameter is $t = 0$.

The tangent vector to the curve is given by

$$\mathbf{r}' = -\sin t\mathbf{i} + \frac{1}{2}\cos t\mathbf{j} - 2\sin t(a+\cos t)\mathbf{k},$$

and at the point P the tangent vector is $\mathbf{r}'(0) = \frac{1}{2}\mathbf{j}$.

2. Compute the work done by the force $\mathbf{F} = -y\mathbf{i} + \mathbf{j}$ along the curve C : $\mathbf{r} = (R\sin t + cRt)\mathbf{i} + (R\cos t + R)\mathbf{j}$ from $(0, 2R)$ to $(2\pi cR, 2R)$. Here, c is a constant.

A: The work is

$$W = \int_C \mathbf{F} \cdot \mathbf{r}' dt = \int_0^{2\pi} (-(R\cos t + R)\mathbf{i} + \mathbf{j}) \cdot ((R\cos t + cR)\mathbf{i} - R\sin t\mathbf{j})\, dt$$

$$= -R^2 \int_0^{2\pi} (\cos t + 1)(\cos t + c) dt - R \int_0^{2\pi} \sin t\, dt,$$

now, $\int_0^{2\pi} \sin t\, dt = 0$, thus

$$= -R^2 \int_0^{2\pi} \left(\cos^2 t + (1 + c)\cos t + c\right)\, dt,$$

and using the identity $\cos^2 t = (1 + \cos 2t)/2$ yields

$$= -R^2 \int_0^{2\pi} \left(\frac{1}{2} + c\right) dt - R^2 \int_0^{2\pi} \left(\frac{1}{2}\cos^2 t + (1 + c)\cos t\right) dt.$$

Since $\int_0^{2\pi} \cos t\, dt = 0$, we finally obtain

$$W = -2\pi R^2 (\frac{1}{2} + c).$$

3. A satellite's orbit is circular at 4040 mi above the earth (earth's radius is 3960 mi), and it completes one revolution every 1000 min. What is the gravity acceleration at its orbit?

A: We may write the satellite's trajectory as

$$\mathbf{r}(t) = R\cos\omega t\, \mathbf{i} + R\sin\omega t\, \mathbf{j},$$

and, then, the acceleration is given by $\mathbf{r}'' = -\omega^2 \mathbf{r}$.

Now, we need only the magnitude of the acceleration, so we compute $a = \|\mathbf{r}''\| = \omega^2 R$. We have that $\|\mathbf{r}\| = R$, and $R = 8000\ mi = 8000 \times 1610\ m$, also $\omega = 2\pi/T = 2\pi/(1000 \times 60)\ 1/sec$.

We conclude that

$$a = \left(\frac{2\pi}{6 \cdot 10^4}\right)^2 \times 8000 \times 1610\ m/sec^2 = 0.14\ m/sec^2.$$

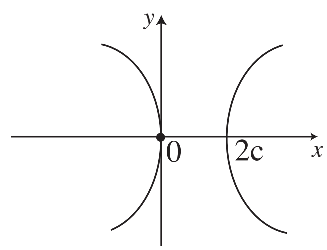
4. Find the **isotherms** (curves of constant temperature) of the planar temperature distribution

$$T(x, y) = \frac{2x}{x^2 - y^2};$$

how many branches does each isotherm have? Find the unit tangents to the curve when $y = 0$?

A: An isotherm is given by $T = c$, where c is a constant. Actually, we choose $T = 1/c$, thus, $2x/(x^2 - y) = 1/c$.

By using simple manipulations we obtain $2cx = x^2 - y^2$, thus, $y = \pm\sqrt{x^2 - 2cx}$, or $y = \pm\sqrt{x(x - 2c)}$. As can be seen in the figure on the right, the curve has two branches. Clearly, the tangents at $y = 0$ is $\pm\mathbf{j}$.



5. Given the function $w = x^2 + 5y^2$ find the integral

$$\oint_C \frac{\partial w}{\partial n}\, ds,$$

where C is the closed curve that encloses the area $x^2 + y^2 = b^2$ and $0 \le y$. Here, $\partial w/\partial n = \mathbf{n} \cdot \nabla w$ is the normal derivative, b is a constant, and $\mathbf{n}$ is the outer unit normal.

A: Green's Theorem can be written as

$$I = \oint_C \frac{\partial w}{\partial n}\, ds = \iint_R \nabla^2 w \, dxdy.$$

Now

$$\nabla^2 w = \nabla \cdot (\nabla w) = \nabla \cdot (2x\mathbf{i} + 10y\mathbf{j}) = 12.$$

We conclude, since the area is one-half of the disc of radius b, that

$$I = \iint_R 12\, dxdy = 6\pi b^2.$$

6. Find the work of the force $\mathbf{F} = e^x\,\mathbf{i} + e^y\,\mathbf{j} + e^z\,\mathbf{k}$ along the curve C given by $x = \ln y$, $z = y^2$, for $1 \le y \le 2$.

A: We have, using y as the parameter,

$$\mathbf{r}(y) = \ln y\,\mathbf{i} + y\,\mathbf{j} + y^2\,\mathbf{k}, \qquad 1 \le y \le 2.$$

When $y = 1$ $\mathbf{r}(1) = (0, 1, 1)$ and when $y = 2$ $\mathbf{r}(2) = (\ln 2, 2, 4)$.

Next, we note that the force is conservative, and

$$\mathbf{F} = \nabla(e^x + e^y + e^z).$$

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = (e^x + e^y + e^z)\big|_{(0,1,1)}^{(\ln 2,2,4)}$$

$$= (2 + e^2 + e^4) - (1 + e + e) = e^4 + e^2 - 2e + 1.$$

7. What kind of surfaces are the level surfaces of the function

$$f(x, y, z) = z^2 - y\ ?$$

Find the unit normal $\mathbf{n}$ to the surface at the point $(4, 6, 0)$.

A: The level surfaces are given by $f = c$, so $z^2 - y = c$, thus,

$$y = z^2 - c.$$

This is a parabola in the yx-plane and extends as a cylinder in the x-direction.

The normal to the surface is given by the gradient $\mathbf{N} = \nabla f = -\mathbf{j} + 2z\mathbf{k}$. At the point $P(4, 6, 0)$ the normal is

$$\mathbf{n} = -\mathbf{j}.$$

8. Compute

$$I = \int_{(0,0,0)}^{(3,2,2)} (z^2\,dy + yz\,dz).$$

A: First, we need to identify the force $\mathbf{F}$ and the curve $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. So, we write

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}, \qquad d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}.$$

Therefore, $\mathbf{F}\cdot d\mathbf{r} = z^2dy + yzdz$, and we conclude that $F_1 = 0$ $F_2 = z^2$ $F_3 = yz$. Next, we need to choose a path from the point $(0,0,0)$ to $(3,2,2)$, and the simplest way is $x = 3t/2$, $y = z = t$, for $0 \le t \le 2$, thus,

$$\mathbf{r}(t) = \frac{3}{2}t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \qquad 0 \le t \le 2.$$

Finally, we obtain

$$I = \int (z^2dy + yzdz) = \int \mathbf{F}\cdot d\mathbf{r} = \int_0^2 (t^2 + t^2)\,dt = \frac{2}{3}t^3|_0^2 = \frac{16}{3}.$$

9. [10pts](You have to answer this question!) Obtain the formula for the area A of a domain R enclosed by the closed curve C,

$$A = \frac{1}{2}\oint_C (x\,dy - y\,dx),$$

from Green's Theorem.

A: Green's Theorem states that if R is a region in the plane and C is a simple curve that bounds it, and if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$, then

$$\oint_C (F_1dx + F_2dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy.$$

(i) Choose $F_1 = 0,\ F_2 = x$, and then $\partial F_2/\partial x = 1,\ \partial F_1/\partial y = 0$. Hence,

$$A = \iint_R (1-0)\,dxdy = \iint_R dxdy = \oint_C xdy.$$

(ii) Choose next $F_1 = -y,\ F_2 = 0$. Then $\partial F_2/\partial x = 0,\ \partial F_1/\partial y = -1$, and so

$$A = \iint_R (0+1)\,dxdy = \iint_R dxdy = \oint_C -y\,dx.$$

Adding now both expression and dividing by 2 yields the result.

10. [10pts](You have to answer this question!) Find the unit normal vector $\mathbf{n}$ of the elliptical cone $z^2 = ax^2 + by^2$ at the point $P(1, 2, \sqrt{a+4b})$.

A: We write

$$f = ax^2 + by^2 - z^2,$$

then $f = 0$ is the elliptical cone. To find the normal we need to compute the normal $\mathbf{N}$,

$$\mathbf{N} = \nabla f = 2ax\mathbf{i} + 2by\mathbf{j} - 2z\mathbf{k}.$$

At the point P we have $x = 1,\ y = 2,\ z = \sqrt{a+4b}$, then

$$\nabla f = 2a\,\mathbf{i} + 4b\,\mathbf{j} - 2\sqrt{a+4b}\,\mathbf{k}.$$

Finally, the unit normal is obtained by dividing by its norm yields

$$\mathbf{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{2a\mathbf{i} + 4b\mathbf{j} - 2\sqrt{a+4b}\,\mathbf{k}}{4a^2 + 16b^2 + 4(a+4b)}.$$